PG (MT) 06 : Group B

Functional Analysis

UNIT 1

(***Contents*** : Metric spaces, metric Topology, convergent and Cauchy sequences, completeness, metric space of all real sequences, complete metric spaces l_p, $C[a,b]$; Metric sub-spaces, separable metric space, continuous functions, Homeomorphism, Isometry, Compact metric spaces, Sequential compactness, Arzela-Ascoli Theorem)

§ 1.1 METRIC SPACES :

Let X be a non-empty set; so the Cartesian product $X_x X$ of all ordered pairs (x, y) of elements $x, y \in X$ is also non-empty.

Definition 1.1.1. A function $d : X_x X \to R$ (reals) is called a metric or a distance function over X if it satisfies following conditions, known as metric or distance axioms :

(M.1) $d(x, y) \geq 0$ for all $x, y \in X$, and $d(x, y) = 0$ if and only if $x = y$. (Property of non-negativity),

(M.2) $d(x, y) = d(y, x)$ for all $x, y \in X$. (Property of symmetry).

(M.3) $d(x, z) \leq d(x, y) + d(y, z)$ for all x, y and $z \in X$. (Property of triangle inequality).

If d is a metric on X, then the pair (X, d) is called a metric space. In a metric space (X, d) if $x_0 \in X$ and r is a +ve real, we have

Definition 1.1.2. The subset $\{x \in X : d(x_0, x) < r\}$ of X denoted by $B_r(x_0)$ is called an open ball in X, centred at x_0 with radius $= r$.

For example, if $d(x, y) = |x - y|$ for any two reals $x, y \in R$, then (R, d) is a metric space and for $x_0 \in R$ and r any +ve r; open ball $B_r(x_0) = \{x \in R : |x - x_0| < r\}$
$= \{x \in R : x_0 - r < x < x_0 + r\}$
= an open interval $(x_0 - r; x_0 + r)$ with mid point x_0 and length $= 2r$.

Similarly, in the metric space ¢ of all complex numbers with usual metric we find an open ball $B_r(z_0)$ looks like an open circular disc with centre at $z_0 \in$ ¢ having radius $= r$.

Definition 1.1.3. The subset $\{x \in X : d(x_0, x) \leq r\}$ of a metric space (X, d) is called a closed ball centred at x_0 with radius $= r$.

The subset $\{x \in X : d(x_0, x) = r\}$ of X is called a sphere centred at x_0 with radius = r. It is also called boundary (Bdr) of open (closed) ball centred at x_0 having radius = r.

The open balls in a metric space (X, d) form a base for a Topology, called metric Topology τ_d (induced by the metric d) on X. So every metric space (X, d) is a topological space with metric topology τ_d. This metric topology τ_d is Hausdorff (T_2).

Definition 1.1.4. A sequence $\{x_n\}$ in (X, d) is said to be a convergent sequence if there is a member $u \in X$ such that, $\lim_{n\to\infty} d(u, x_n) = 0$.

Or, equivalently, given any +ve ε, there is an index N such that $d(u, x_n) < \varepsilon$, when $n \geq N$.

If $\{x_n\}$ is a convergent sequence in (X, d) with $u \in X$ and $\lim_{n\to\infty} d(u, x_n) = 0$, we write $\lim_n x_n = u \in X$, and u is a unique member of X, because metric space is Hausdorff.

Definition 1.1.5. A sequence $\{x_n\}$ is said to be a Cauchy sequence in (X, d) if $d(x_n, x_m) \to 0$ as $n, m \to \infty$.

Or, equivalently, given any +ve ε, there is an index N satisfying $d(x_n, x_m) < \varepsilon$ whenever $n, m \geq N$.

It is an easy exercise to see that in a metric space every convergent sequence is cauchy, but converse is false.

Definition 1.1.6. A metric space (X, d) is said to be complete if every Cauchy sequence in (X, d) is convergent in X.

For example, real number space R with usual metric $d(x, y) = |x - y|; x, y \in R$ is a complete metric space. This is what is known as Cauchy's General Principle of convergence; and essentially by same reason the Euclidean n-space R^n consisting of all n tuples of reals like $\underset{\sim}{x} = (x_1, x_2, ..., x_n), x_i \in R$ is also a complete metric space with usual/Euclidean metric d where $d^2(\underset{\sim}{x}, \underset{\sim}{y})$

$$= \sum_{i=1}^{n} |x_i - y_i|^2; \underset{\sim}{x} = (x_1, x_2, ..., x_n),\ \underset{\sim}{y} = (y_1, y_2, ..., y_n) \in R^n$$

Example 1.1.1. The collection S of all sequences of reals is a complete metric space with metric $\rho(\underset{\sim}{x}, \underset{\sim}{y}) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|\xi_i - \eta_i|}{1 + |\xi_i - \eta_i|}$, where $\underset{\sim}{x} = (\xi_1, \xi_2, ...)$,

$\underset{\sim}{y} = (\eta_1, \eta_2, ...) \in S$. The r.h.s. series is convergent because each term is domoninated by a corresponding term of a convergent geometric series. Here is a routine exercise to see that metric axioms are all satisfied. For completeness part we remark on passing that if $a_{n,m} \geq 0$, then $a_{n,m} \to 0$ if and only if $\dfrac{a_{n,m}}{1+a_{n,m}} \to 0$ as $n, m \to \infty$.

Take $\{\underset{\sim}{x}_n\}$ as a Cauchy sequence of elements in S

where $\underset{\sim}{x}_n = (\xi_1^{(n)}, \xi_2^{(n)}, ... \xi_i^{(n)}, ...)$.

Corresponding to a + ve ε we find an index N such that

$\rho(\underset{\sim}{x}_n, \underset{\sim}{x}_m) < \varepsilon$ for all $n, m \geq N$

or, $\displaystyle\sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|\xi_i^{(n)} - \xi_i^{(m)}|}{1+|\xi_i^{(n)} - \xi_i^{(m)}|} < \varepsilon$ for all $n, m \geq N$ .... (1.1.7)

As individual term in series above is ≥ 0, we appeal to the remark made earlier to say that $|\xi_i^{(n)} - \xi_i^{(m)}| \to 0$ as $n, m \to \infty$. And hence for each co-ordinate i by Cauchy's General Principle of Convergence, $\{\xi_i^{(n)}\}$ is convergent.

Put $\lim\limits_{n\to\infty} \xi_i^{(n)} = \xi_i^{(0)}$, i = 1, 2, ...

Taking $\underset{\sim}{x}_0 = \left(\xi_1^{(0)}, \xi_2^{(0)}, ...\right)$ we find $\underset{\sim}{x} \in S$ and passing on limit as $m \to \infty$ in (1.1.7) we have

$$\sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|\xi_i^{(n)} - \xi_i^{(0)}|}{1+|\xi_i^{(n)} - \xi_i^{(0)}|} \leq \varepsilon \text{ for } n \geq N.$$

That means, $\lim\limits_{n\to\infty} \rho(\underset{\sim}{x}_n, \underset{\sim}{x}_0) = 0$

or $\lim\limits_{n\to\infty} \underset{\sim}{x}_n = \underset{\sim}{x}_0 \in S$

So the sequence space S becomes a complete metric space.

Remark : The convergence of sequence of elements in S as shown above is known as co-ordinatewise convergence; that is to say, $\lim\limits_{n\to\infty} \underset{\sim}{x}_n = \underset{\sim}{x}_0$ in S,

where $\underset{\sim}{x}_n = \left\{\xi_i^{(n)}\right\}$ and $\underset{\sim}{x}_0 = \left\{\xi_i^{(0)}\right\}$, if and only if $\lim\limits_{n\to\infty} \xi_i^{(n)} = \xi_i^{(0)}$, for i = 1, 2, 3, ...; **The convergence is not necessarily uniform.**

Example 1.1.2. The sequence space l_p ($1 < p < \infty$) consisting of all sequences $\underset{\sim}{x} = (\xi_1, \xi_2, ..., \xi_n, ...)$ of reals with $\sum_{i=1}^{\infty} |\xi_i| p < +\infty$ is a complete metric space with metric $\rho(\underset{\sim}{x}, \underset{\sim}{y}) = \left(\sum_{i=1}^{\infty} |\xi_i - \eta_i| p \right)^{1/p}$, for $\underset{\sim}{x} = (\xi_1, \xi_2, ...), \underset{\sim}{y} = (\eta_1, \eta_2, ...) \in l_p$.

Example 1.1.3. The function space $C[a,b]$ consisting of all real valued continuous functions over the closed interval $[a, b]$ is a complete metric space with sup metric $\rho(f, g) = \sup_{a \le t \le b} |f(t) - g(t)|$, as $f, g \in C[a,b]$

The last two examples appear in Book PGMT 2A. They are referred to there.

§ 1.2 SUB-SPACES :

Let Y be a non-empty subset of a metric space (X, d). There is a natural metric, namely the restriction d_Y of d to $Y \times y$.

Definition 1.2.1. The metric space (Y, dy) is called a sub-space of (X, d).

Theorem 1.2.1. A subset A in Y is open in (Y, dy) if and only if there is a subset A_1 in X that is open in (X, d) such that $A = Y \cap A_1$.

Proof : Let $x \in X$ and $y \in Y$ and r be a +ve number n and let $B_X(x, r)$ and $B_Y(y, r)$ denote open balls centred at x and at y respectively with radius = r in (X, d) and in (Y, dy).

Then we have $B_Y(y,r) = Y \cap B_x(y,r)$ for all $y \in Y$, and $r > 0$ ....(1.2.1)

Take A as an open set in (Y, d_Y), then we know that A is a Union of some open balls of (Y, d_Y); say of $\{B_Y(y,r)\}$ as $y \in A$ and $r > 0$.

Thus $A = UB_Y(y,r)$

$= U\{Y \cap B_X(y,r)\}$ by (1.2.1)

$= Y \cap \{UB_X(y,r)\}$

$= Y \cap A_1$ (say)

where A_1 is a union of open balls in (X, d) and A_1 is an open set in (X, d).

Conversely, let $A = Y \cap A_1$, where A_1 is an open set in (X, d). For $y \in A$, there is an open ball $B_X(y,r) \subset A_1$, and hence $B_Y(y,r) = Y \cap B_X(y,r) \subset (Y \cap A_1) = A$. So every member of A attracts an open ball in (Y, d_Y) *i.e.* A is an open set in (Y, d_Y). The proof is complete.

Corollary : A is closed in (Y, d_Y) if and only if there is a subset A_1 of X that is closed in (X, d) such that $A = Y \cap A_1$. (If $A = Y \cap A_1$, we have $Y \setminus A = Y \cap (X \setminus A_1)$, and now proceed).

Definition 1.2.2. A metric space (X, d) is said to be separable if and only if there is a countable subset D of X such that D is dense in (X, d) (or equivalently, $\overline{D}$ (closure of D) = X).

For example, real number space R with usual metric is separable, because the set Q of all rationals if R is dense in R, where we know that Q is countable.

Theorem 1.2.2. A sub-space of a separable metric space is separable.

Proof : Let (Y, d_Y) be a sub-space of (X, d) which is a separable metric space. Let $A = \{x_1, x_2, ..., x_n, ...\}$ be a countable set in X such that $\overline{A} = X$.

If $y \in Y$, then for each +ve integer m the open ball $B\left(y, \frac{1}{m}\right)$ meets A at some point, say = x_n.

Thus $x_n \in \left\{A \cap B\left(y, \frac{1}{m}\right)\right\}$.

So, Open ball $B\left(x_n, \frac{1}{m}\right) \cap Y \neq \phi$

Put $\Delta = \left\{(n, m) : B\left(x_n, \frac{1}{m}\right) \cap Y \neq \phi\right\}$. Thus $\Delta \neq \phi$. For each $(n, m) \in \Delta$, take a member $y_{n,m} \in \left\{B\left(x_n, \frac{1}{m}\right) \cap Y\right\}$, and put $B = \left\{y_{n,m} : (n, m) \in \Delta\right\}$. Therefore B is a countable subset of Y because Δ is so. We now verify that B is dense in (Y, d_Y). Take $y \in Y$ and $r > 0$; choose +ve integer m so that $\frac{1}{m} \leq \frac{1}{2} r$. As said above there is an integer n such that $x_n \in B\left(y, \frac{1}{m}\right)$. Then $(n, m) \in \Delta$, and we have

$$d(y, y_{n,m}) \leq d(y, x_n) + d(x_n, y_{n,m}) < \frac{1}{m} + \frac{1}{m} = \frac{2}{m} \leq r.$$

That means, $y_{n,m} \in B(y, r)$. Therefore $y \in \overline{B}$ in (Y, d_Y), or, B is dense in (Y, d_Y).

§ 1.3 CONTINUOUS FUNCTIONS :

Let (X, d) and (Y, ρ) be two metric spaces.

Definition 1.3.1. A function $f:(X,d)\to(Y,\rho)$ is said to be continuous at a point $c\in X$, if and only if given a +ve ε, there is a +ve δ (depending on ε and c) such that $e(f(x),f(c))<\varepsilon$ whenever $d(x,c)>\delta$.

or equivalently, $f(B(c,\delta))\subset B(f(c),\in)$.

f is said to be a continuous function if f remains continuous each point of X.

Further details on continuous functions over metric spaces may be seen in Book PGMT 2A.

Homeomorphism, Isometry :

Definition 1.3.1. A function $f:(X,d)\to(Y,\rho)$ is said to be a homeomorphism if f is 1–1, onto (bijective) and both f and f^{-1} are continuous functions.

If there is a homeomorphism between (X, d) and (y, ρ), then two metric spaces (X, d) and (y, ρ) are called homeomorphic.

Explanation : If f is a homeomorphism of X onto Y, then f^{-1} is so between Y and X. Also it is a routine matter to see that composition of two homeomorphisms is again a homeomorphism; thus in the family of all metric spaces the binary relation 'of being homeomorphic' is an equivalence relation.

Example 1.3.1. Consider the metric space R of reals with usual metric and a function $T:R\to R$ given by $T(x)=x+a$, where a is a fixed real number, and $x\in R$. Then this translation function (equals to Identity function when a = 0) is a homeomorphism; here $T^{-1}:R\to R$ is given by $T^{-1}(x)=x-a;\ x\in R$. Similarly one shows that for any non-zero real λ, multiplication function $M_\lambda:R\to R$ given by $M_\lambda(x)=\lambda x, x\in R$ is a homeomorphism, where $M_\lambda^{-1}=M_{\lambda^{-1}}$.

We know that family of all open sets in (X, d) forms a Topology, called metric topology τ_d on X induced by d. Any property in a metric space (X, d) that can be formulated entirely in terms of members of τ_d (open sets) is known as **a Topological property.**

Consequently, homeomorphic metric spaces have the same topological properties like convergence of sequences in the space and continuity of functions over the space. Following example shows **completness is not a topological property in a metric space.**

Example 1.3.1. Take $X = \{1, 2, 3, ...\}$ and $Y = \{1, \frac{1}{2}, \frac{1}{3}, ...\}$. Regarded as a subspace of the space R of reals with usual metric we find that spaces X and Y are discrete metric spaces (every subset being both open and closed); thus the function $h: X \to Y$ where $h(n) = n^{-1}$ is a homeomorphism of X onto Y. Since X is a closed subset of R which is a complete metric space, the space X is complete. On the other hand Y is not complete.

Definition 1.3.2. A function $f: Y \to Y$ that is onto (surjective) is said to be an Isometry if $e\left(f(x), f(y)\right) = d(x, y)$ for all $x, y \in X$.

Explanation : Identity function on X is an Isometry of X onto itself. Also a transformation of rotation like $x' = x\cos\theta + y\sin\theta$, $y' = -x\sin\theta + y\cos\theta$ is an Isometry of Eudidean 2-space R^2 onto itself with usual metric. Also an Isometry is a homeomorphism. Thus two metric spaces that are isometric are indistinguishable in respect of their metric properties.

Example 1.3.2. In metric space (X, d) take $x_0 \in X$.

For $x \in X$, Let $f_x : x \to R$ (space of reals with usual metric) be given as

$$f_x(y) = d(y, x) - d(y, x_0) \text{ for } y \in X.$$

Then show that $x \to f_x$ is an isometry of X into $C(X)$ where $C(X)$ is metric space of all real valued continuous functions over X with sup metric

$$\| f - g \| = \sup_{y \in X} | f(y) - g(y) | < \infty.$$

As distance function d is continuous, it follows that f_x is continuous for all $x \in X$.

Solution : Take $u, v \in X$; so we have

$$\left.\begin{array}{l} f_u(y) = d(y, u) - d(y, x_0) \\ \text{and } f_v(y) = d(y, v) - d(y, x_0) \quad \text{for all } y \in X \end{array}\right\} \quad \text{.....................................(1.3.1)}$$

So, $| f_u(y) - f_v(y) | = | d(y, u) - d(y, v) | \leq d(u, v)$ which is independent of $y \in X$; taking the $\sup_{y \in X}$ over L.H.S. we obtain

$$\sup_{y \in X} | f_u(y) - f_v(y) | \leq d(u, v)$$

or $\| f_u - f_v \| \leq d(u, v)$ ....1.3.2)

Putting $y = u$ in (1.3.1) we have,

$$f_u(u) = -d(y, x_v) \text{ and } f_v(u) = d(u, v) - d(y, x_0)$$

So, $| f_u(u) - f_v(u) | = d(u, v)$

Now $\sup_{y \in X} | f_u(y) - f_v(y) | \geq | f_u(u) - f_v(u) | = d(u, v)$ .... (1.3.3)

from (1.3.2) and (1.3.3) we obtain

$$\| f_u - f_v \| = d(u, v).$$

Thus $x \to f_x$ invites an Isometry of X into $C(X)$.

§ 1.4 COMPACT METRIC SPACES :

Some important properties of reals as we encounter in real analysis had motivated more important concepts in a metric space like completeness and compactness. Cauchy's General Principle of Convergence is the driving force behind completeness in a metric space. Essence of Heine-Borel Theorem could be found in concept of compactness in a metric space.

In consequence, it had been an inevitable task with urgency to identify compact subsets in a metric space. Russian Mathematicians like Alexandrov and Urysohn had been responsible to put forward notion of compactness via 'open cover' in the space; on the other hand close to Bolzano-Weirstrass property is classical analysis concept of sequential compactness owed to Frechet in a metric space. And now we know for certain that these two routes are equivalent in describing compactness in a metric space. For details in this context see the book PGMT 2A.

It has been possible to discover that a subset in Euclidean n-space R^n with usual metric is compact if and only if the subset is a bounded and closed set in R^n.

Given a metric space X it is often hard to decide which subsets of X are compact, and which are not. Our present task is the job of identifying compact subsets of a very important and useful function space of some continuous functions that we presently discuss below. The concerned target theorem in this connection is Ascoli-Arzela Theorem.

Definition 1.4.1. Let (X, d) denote a metric space.

(a) A family $Q = \{A_1\}_{i\in\Delta}$ of open sets A_i in (X, d) is said to be an open cover for X if every element of X belongs to at least one member A_i of the family Q. That is to say, $X \subset \underset{i\in\Delta}{U} A_i$.

(b) A sub-family of an open cover for X which by itself is an open cover for X is called sub-cover for X.

(c) (X, d) is said to be a compact metric space if every open cover for X has **a finite sub-cover for *X*.**

Explanation : By a finite sub-cover we mean that the sub-cover consists of a finite number of members only. Consider a family $\{(-n, n)\}_{n\in N}$ (N = set of all natural numbers). Its members are open intervals, and hence open sets in the metric space R of reals with usual metric. It is an open cover for R; because $R = \underset{n=1}{U}(-n, n)$. Clearly, this open cover possesses no finite sub-cover for R. That is why, R is not compact.

Definition 1.4.2. A subset G of (X, d) is said to be compact if as a sub-space of (X, d) it is compact under definition 1.4.1.

For example, although R is not compact with usual metric any finite subset of R becomes compact.

Definition 1.4.3. (X, d) is said to be sequentially compact if every sequence in X has a convergent sub-sequence in X.

It is a bit lengthy exercise to conclude that a metric space is compact if and only if it is a sequentially compact. See book PGMT 2A.

The function space *C*[*a*, *b*] of all real-vaheed continuous functions over a closed interval [*a*, *b*].

We know that $C[a, b]$ is a complete metric space with respect to sub metric $\rho(f, g) = \sup_{a\le t\le b} |f(t) - g(t)|$, $f, g \in C[a, b]$. But $C[a, b]$ is not compact with respect to sub metric, because $C[a,b]$ is not bounded; for all constant functions like $f_n(t) = n$ for $a \le t \le b$ satisfy $\rho(f_n, 0) = n \to \infty$ as $n \to \infty$. However there are compact sets in $C[a, b]$. In searching then we need some Definitions.

Definition 1.4.1. (a) A subset M of $C[a, b]$ is said to be **uniformly bounded** if there is a +ve constant K such that $|x(t)| \le K$ for all t in $a \le t \le b$ and for all members $x \in M$.

(b) Subset M is said to be **equi-continuous** if given any +ve ε, there is a +ve δ (depending on ε only) such that $|x(t_1) - x(t_2)| < \varepsilon$ whenever $|t_1 - t_2| < \delta (t_1, t_2 \in [a,b])$ for all members $x \in M$.

Example 1.4.1. Show that the subset $\{f_n\} \subset C[0,1]$ is equibounded where $f_n(t) = 1 + \frac{t}{n}$; $0 \le t \le 1$.

Solution : Here $|f_n(t)| = |1 + \frac{t}{n}| \le 1 + |\frac{t}{n}| \le 1 + \frac{1}{n} \le 2$ for all n and for all t in $0 \le t \le 1$. So the conclusion stands.

Theorem 1.4.1 (Arzela-Ascoli Theorem) : A subset M of $C[a,b]$ is compact if and only if M is uniformly bounded and equi continuous.

Proof : The condition is necessary : Let M be a compact subset of $C[a,b]$ (*w.r.t.* sup metric). Then M is bounded, because a compact set in a metric space is bounded and closed. Thus we find a closed ball say $\bar{B}_r(x_0)$ centred at $x_0 \in C[a,b]$ with radius = r, such that

$$M \subset \bar{B}_r(x_0)$$

Thus $\sup_{a \le t \le b} |x(t) - x_0(t)| \le r$

Now $x(t) = x(t) - x_0(t) + x_0(t)$ and

$$\sup_{a \le t \le b} |x(t)| \le \sup_{a \le t \le b} |x(t) - x_0(t)| + \sup_{a \le t \le b} |x_0(t)| \le r + k, \text{ say,}$$

where $k = \sup_{a \le t \le b} |x_0(t)|$.

That means $|x(t)| \le (r + R) = K$ (say) for all t in $a \le t \le b$ and for all $x \in M$. Hence M is uniformly bounded.

For equi-continuity take a +ve ε.

Since M is compact, we find an $\frac{\varepsilon}{3} - net = (x_1(t), x_2(3), ..., x_n(t))$ for M.

Since every real-valued continuous function over a closed interval is uniformly continuous. So here each of the members $x_1, x_2, ...x_k$ of $C[a,b]$ is uniformly continuous in $[a,b]$.

So, for each $x_i(t)$ we find a +ve δ_i such that

$$|x_i(t_1)-x_i(t_2)|<\tfrac{\varepsilon}{3} \quad \text{whenever } |t_1-t_2|<\delta_i,\ t_1,t_2\in[a,b].$$

Now take a +ve δ = $\min\limits_{1\le i\le k}\{d_i\}$. Then we have

$$|x_i(t_1)-x_i(t_2)|<\frac{\varepsilon}{3} \text{ whenever } |t_1-t_2|<\delta, t_1,t_2\in[a,b] \text{ for all } i = 1,2,\ldots\ldots, k.$$

Now for every member $x \in M$, we find a member, say, x_i from $\frac{\varepsilon}{3}$–net, such that

$$\rho(x,x_i)<\frac{\varepsilon}{3} \quad (\rho = \text{sup-metric of } C[a,b]).$$

If $t_1,t_2\in[a,b]$ and $|t_1-t_2|<\delta$ we have

$$|x(t_1)-x(t_2)|\le x(t_1)-x_i(t_1)|+|x_i(t_1)-x_i(t_2)|+|x_i(t_2)-x(t_2)|$$

$$\le \sup_{a\le t\le b}|x(t)-x_i(t)|+|(x_i(t_1)-x_i(t_2)|+\sup_{a\le t\le b}|x_i(t)-x(t)|$$

$$<\rho(x,x_i)+\frac{\varepsilon}{3}+\rho(x,x_i)<\varepsilon\,.$$

This inequality holds for all $t_1,t_2\in[a,b]$, with $|t_1-t_2|<\delta$ and for all members $x\in M$. So M is equi-continuous.

The condition is sufficient : Suppose M is uniformly bounded and equi-continuous ; we show that M is compact. Because $C[a,b]$ is complete and so is M; It suffices to show that every sequence in M has a Cauchy subsequence. Let $D = (t_2, t_3, t_4, \ldots\ldots)$ be a countable dense set of reals in $[a,b]$.

Suppose $S_1=(f_{11},f_{12},f_{13},\ldots\ldots)$ be any sequence of elements in M. By uniform boundedness property of M. We find a +ve K such that

$|f(t)|\le K$ for all t in $a\le t\le b$ and for all $f\in M$. .... (1.4.6)

Let us examine real sequence

$$\{f_{11}(t_2),\ f_{12}(t_2),\ f_{13}(t_2),\ldots,\ f_{1m}(t_2),\ldots\}$$

From (1.4.6) it is clear that this is a bounded sequence of reals and has a convergent subsequence.

Let $S_2=(f_{21},\ f_{22},\ f_{23},\ldots\}$ be a sub-sequence of S_1 above such that $\{f_{21}(t_2),\ f_{22}(t_2),\ f_{23}(t_2),\ldots\ldots\}$ converges.

Now examine real sequence $\{f_{21}(t_3), f_{22}(t_3), f_{23}(t_3),.....\}$, and by similar reasoning as above, we have

$S_3 = \{f_{31}, f_{32}, f_{33},.....\}$ as a subsequence of S_2 such that

$\{f_{31}(t_3), f_{32}(t_3), f_{33}(t_3),.....\}$ is convergent.

We continue this chain to construct S_1, S_2, S_3, ... of sequences of functions like :

$$S_1 = \{f_{11}, f_{12}, f_{13},.....\}$$
$$S_2 = \{f_{21}, f_{22}, f_{23},.....\}$$
$$S_3 = \{f_{31}, f_{32}, f_{33},.....\}$$
$$.... \quad .... \quad ....$$

where S_m constitutes a subsequence of $S_{m-1}(m = 2,3,....)$ with the property that $\{f_{n1}(t_n), f_{n2}(t_n)\ f_{n3}(t_n), ...\}$ is a convergent sequence of reals.

Now put $f_n = f_{nn}$ $(n = 2, 3, 4,...)$ then $\{f_1, f_2, f_3,....\}$ is the diagonal subsequence of S_1. From mode of construction

$x_n \in D$ and $\{f_1(t_n), f_2(t_n), ..., f_i(t_n),...\}$ is a convergent real sequence.

If $i > k$, consider $|f_{ii}(t_n) - f_{kk}(t_n)|$ for $i > k > n$ and knowing that both $f_{ii}(t_n), f_{kk}(t_n)$ are members of convergent real sequence

$$\{f_{n1}(t_n), f_{n2}(t_n), (f_{n3}(t_n), ...\}$$

We have $|f_i(t_n) - f_k(t_n)| \to 0$ as $i, k \to \infty$. Thus $\{f_1(t_n), f_2(t_n), f_3(t_n),....\}$ is a Cauchy sequence of reals.

Finally, take any +ve ε. Since M is equi-continous and $S \subset S, \subset M$, we find a +ve δ such that $|f_n(t) - f_n(t') < \frac{\varepsilon}{3}$ whenever $|t - t'| < \delta$, $t, t' \in [a,b]$ for all members $f_n \in S$.

Now consider the family $\{t_n - \delta, t_n + \delta)\}$ of open intervals with mid point $t_n \in D$.

It is routine verification with dense property of D in $[a,b]$ that this family of open intervals becomes an open cover for $[a,b]$. By compactness of $[a,b]$ we obtain a finite sub-over, say

$[a,b] = \underset{t_n \in D}{U}(t_n - \delta, t_n + \delta)$ and $2 \le n \le n_0$

Again $\{f_1(t_n), f_2(t_n), ...\}$ is Cauchy; thus a +ve integer K_0 is there such that

$$| f_i(t_n) - f_k(t_n) | < \frac{\varepsilon}{3} \text{ for all } 2 \le n \le n_0$$

If t is any position of [a, b], we find n with $2 \le n \le n_0$ so that $t_n - \delta < t < t_n + \delta$ and for $i, K \ge K_0$ we have

$$| f_i(t) - f_k(t) | \le | f_i(t) - f_i(t_n) | + | f_i(t_n) - f_k(t_n) |$$

$$+ | f_k(t_n) - f_k(t) | < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

That means $\sup\limits_{a \le t \le b} | f_i(t) - f_k(t) | \le \varepsilon$ for $i, k \ge k_0$

or, $\rho(f_i, f_k) \le \in$ for $i, k \ge k_0$

or, $S = \{f_1, f_2, ...\}$ is a Cauchy subsequence of S_1.

The proof is now complete.

EXERCISE A

Short-answer type questions :

1. Show that compactness is not a heriditary property in a metric space.
2. Give an example to show that a closed bounded set in a metric space may not be compact.
3. Show that $f(x) = x + a$ or $= - x + a$ where a is a fixed real is an Isometry on the space R of reals with usual metric.
4. Prove that any bounded sequence of reals has a convergent subsequence.
5. In a metric space (X, d) if $\lim\limits_{n \to \infty} x_n = x \in X$, show that $\{x_n\} \cup \{x\}$ is copact.

EXERCISE B

Broad questions

1. Show that the closed ball $\tilde{B} = \left\{ x : \sup\limits_{0 \le t \le 1} | x(t) \le 1 \right\}$ of C[0, 1] with supmetric is not compact.

2. Prove that only Isometries of the space R of reals with usual metric are $f(x) = x + a$ and $f(x) = -x + a$ where a is a real number.
3. Give an example of a Homeomorphism that is not an Isometry.
4. Let f be a real-valued function on a compact metric space (X, d), show that f assumes its maximum and minimum on X.
5. Verify that closed Unit ball in sequence space l_2 is bounded without being totally bounded.
6. Let X denote the metric space of all real polynomials $p(t)$ in $0 \leq t \leq 1$; show that X is not a complete metric space with respect to sup metric.