UNIT 2

(***Contents*** : Linear spaces, Dimension of a linear space, Normed linear space (NLS), Banach space, $C[a,b]$ as a Banach space, Quotient space of a NLS, Convex sets, their algebra, Bounded linear operator; its continuity, Unbounded linear operator, Norm ||T|| of a bounded linear operator T; Formulae for ||T||.)

§ 2.1 LINEAR SPACES

Definition 2.1.1. Let R (q) denote the field of reals (complex numbers) that are also called scalars. A linear space (Vector space) V is a collection of objects called vectors satisfying following conditions :

I. V is additively an Abelian (commutative) Group, the identity element of which is called the Zero vector denoted by 0.

II. For every pair (α v), α being a scalar and $v \in V$, there is a vector, denoted by $\alpha.v$ (**not** $v\alpha$), called a scalar multiple of v such that

(a) $1.v = v$ for all $v \in V$.

(b) $\alpha.(u+v) = \alpha.u + \alpha.v$ for all scalars α and for all vectors $u, v \in V$.

(c) $(\alpha+\beta).v = \alpha.v + \beta.v$ for all scalars α and β and for all vectors $v \in V$.

(d) $\alpha.(\beta.v) = (\alpha.\beta).v$ for all scalars α and β and for all $v \in V$.

Example 2.1.1. Let R^n be the collection of all n tuples of reals like $x = (x_1, x_2, ...x_n)$; x_i being reals. Then R^n becomes a linear space with real scalar field where addition of vectors and scalar multiplication of vectors are defined as

$$x + y = (x_1, x_2, ..., x_n) + (y_1, y_2, ..., y_n) = (x_1 + y_1, x_2 + y_2, ..., x_n + y_n) \text{ and}$$

$\alpha.x = \alpha(x_1, x_2, ..., x_n) = (\alpha x_1, \alpha x_2, ...\alpha x_n)$; $x, y \in R^n$ and α any real scalar.

Here R^n is also called Euclidean n-space with the zero vector $\underline{0} = (0, 0, ..., 0)$ (all co-ordinates are zero), and it is a real Linear space.

Example 2.1.2. Let $C[a,b]$ denote the collection of all real valued continuous functions over a closed interval $[a,b]$. Then $C[a,b]$ is a real linear space (associated scalar field being that of reals) where vector sum and scalar multiplication are defined as under :

$$(f+g)(t) = f(t) + g(t)\ ; \quad a \le t \le b, \text{ and } f, g \in C[a,b]$$

and $(\alpha f)(t) = \alpha f(t)$; $a \le t \le b$ and α any real scalar.

As we know that sum of two continuous functions is a continuous function and so is a scalar multiple of a continuous function, we see that $f+g$ and αf are members of $C[a,b]$ where $f, g \in C[a,b]$ and α is any scalar. Here the zero vector equals to the zero function $\left(0(t) = 0; a \le t \le b\right)$ over the closed interval $[a,b]$.

There are many other linear spaces like the sequence spaces $l_p (1 < p < \infty)$, polynomial space $\rho[a,b]$, function space $L_2[a,b]$, that we encounter in our discussion to follow.

Definition 2.1.2. (a) If A and B are subsets of a linear space V then $A + B = \{a + b : a \in A \text{ and } b \in B\}$.

(b) For any scalar λ,

$$\lambda A = \{\lambda a : a \in A\}$$

The subset $A - B = A + (-1)B$; and taking λ = zero scalar we find $0A = \{\underline{0}\}$. Further we see that A + B = $B + A$, because vector addition is commutative, However $A - B \ne B - A$. Taking A and B as singleton and $A = \{(1,0)\}, B = \{(0,0\}$ in Euclidean 2-space R^2, we find $A - B = \{(1,0)\}$ and $B - A = \{(-1,0)\}$.

Further for any scalar α we have $\alpha A = \{\alpha a : a \in A\}$.

Here is a caution. In general, $A + A \ne 2A$.

Because take $A = \{(1,0), (0,1)\}$; Then we have

$2A = \{(2,0), (0,2)\}$ which is not equal to $A+A$

where $A + A = \{(2,0), (0,2), (1,1)\}$.

Given a fixed member $a \in V$, the subset $a + B = \{a + b : b \in B\}$ is called a translate of B.

§ 2.2. Let X denote a linear space over reals/complex scalars. Given $x_1, x_2, \ldots, x_n$ in X, and $\alpha_1, \alpha_2, \ldots, \alpha_n$ as scalars, the vector $\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n$ of X is called a linear combination of $x_1, x_2, \ldots, x_n$.

A subset E of X is said to span (generate) X if and only if every member of X is a linear combination of some elements of E.

Elements $x_1, x_2, \ldots, x_n$ of E are said to be linearly dependent if and only if there are corresponding number of scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$ not all zero such that

$$\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n = \underline{0}$$

A finite number of elements $x_1, x_2, ..., x_k$ of X are said to be linearly independent if they are not linearly dependent. This amounts to say that if

$$\sum_{i=1}^{k} \alpha_i x_i = \underline{0} \text{ implies } \alpha_1 = \alpha_2 = ... = \alpha_k = 0.$$

An arbitrary system of elements of X is called linearly independent if every finite subset of the given system becomes linearly independent.

Observe that if a set of vectors in X contains a linearly dependent subset, whole set becomes linearly dependent. Also note that a linearly independent set of vectors does not contain the zero vector.

Definition 2.2.1. A non-empty sub-set L of a linear space X is called a sub-space of X if $x + y$ is in L whenever x and y are both in L, and also αx is in L, whenever x is in L and α is any scalar.

Example 2.2.1. Let S be any non-empty subset of X. Let L = the set of all linear combinations of elements of S. Then L is sub-space of X, called the sub-space spanned (generated) by S.

The subset = $\{\underline{0}\}$ is a sub-space, called the Null-space.

Theorem 2.2.1. Let $x_1, x_2, ..., x_n$ be a set of vectors of X with $x_1 \neq \underline{0}$. This set is linearly dependent if and only if some one of vectors $x_2, ..., x_n$, say x_k is in the sub-space generated by $x_1, x_2, ..., x_{k-1}$.

Proof : Suppose the given set of vectors is linearly dependent. There is a smallest k with $2 \leq k \leq n$ such that $x_1, x_2, ..., x_k$ is linearly dependent; and we have $\alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_k x_k = 0$ with not all α's are zero scalars. Necessarily, we have $\alpha_k \neq 0$; otherwise $x_1, x_2, ..., x_{k-1}$ would form a linearly dependent set.

In consequence $x_k = -\dfrac{\alpha_1}{\alpha_k} x_1 - \dfrac{\alpha_2}{\alpha_k} x_2 - .... - \dfrac{\alpha_{k-1}}{\alpha_k} x_{k-1}$.

That means x_k is in the sub-space generated by $x_1, x_2, ..., x_{k-1}$.

Conversely, if one assumes that some x_k is in the sub-space generated by $x_1, x_2, ..., x_{k-1}$; then we have

$$x_k = \beta_1 x_1 + \beta_2 x_2 + ... + \beta_{k-1} x_{k-1}$$

That means $x_1, x_2, ..., x_k$ are linearly dependent, and in turn we have the set $(x_1, x_2, ..., x_k)$ as linearly dependent.

Definition 2.2.2. In a linear space *X* suppose there is a +ve integer *n* such that *X* contains **a set of *n* vectors** that are **linearly independent,** while **every set of *n* + 1** vectors in *X* is **linearly dependent,** then *X* is called finite dimensional and ***n*** is called **dimension of *X*** {Dim(*X*)}.

The Null-space is finite dimensional of **dimension 0.**

If *X* is **not** finite dimensional it is called **infinite dimensional.**

Definition 2.2.3. A finite set *B* in linear space *X* is called a basis of *X* if *B* is linearly independent, and *f* the sub-space spanned (generated) by *B* is all of *X*.

Explanation : If $x_1, x_2, ..., x_n$ is a basis for *X*, every member $x \in X$ can be expressed as $x = \alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_n x_n$ where scalar coefficients α_i's are uniquely determined; so *x* does not have a **different** linear combination of basis members.

Suppose $\text{Dim}(X) = n \ (n \geq 1)$. Then *X* has a basis consisting of *n* members; For, *X* certainly contains vectors $x_1, x_2, ..., x_n$ that form a linearly independent set. Now for any member $x \in X$, the set of vectors $x_1, x_2, ..., x_n$ plus *x u* of *n* + 1 vectors must be linearly dependent. Now Theorem 2.2.1 applies to conclude that *x* is in the sub-space generated by $x_1, x_2, ..., x_n$. Hence $x_1, x_2, ..., x_n$ form a basis of *X*.

§ 2.3 NORMED LINEAR SPACES :

Definition 2.3.1. A linear space *X* is called a Normed Linear Space (*NLS*) if there is a non-negative real valued function denoted by || ||, called a norm on *X* whose value at $x \in X$ denoted by $\| x \|$ satisfies following conditions (N.1) – (N.3), called norm axioms :–

(N.1) $\| x \| \geq 0$, and $\| x \| = 0$ if and only if $x = \underline{0}$.

(N.2) $| \alpha x \| = | \alpha | \| x \|$ for any scalar α and for any $x \in X$.

(N.3) $\| x + y \| \leq \| x \| + \| y \|$ for any two members *x* and *y* in *X*.

If || || is a norm on *X*, the ordered pair (*X*, || ||) is designated as a *NLS*. If norm changes, *NLS* also changes.

In a *NLS* (*X*, || ||) one can define a metric ρ by the rule : $\rho(x, y) = \| x - y \|$ for all $x, y \in X$. It is an easy task to check that ρ satisfies all metric axioms; and (*X*,ρ) becomes **a metric space** with the metric topology called **Norm Topology** because of its induction from norm || ||. We write $\lim_{n\to\infty} x_n = x$ in X off $\lim_{n\to\infty} \| x_n - x \| = 0$; this convergence in *NLS X* is known as convergence in Norm. Similarly, we define a Cauchy sequence in *NLS X*.

A subset B in a *NLS X* is said to be bounded if there is a +ve K such that $\| x \| \leq K$ for all $x \in B$

Let $x_0 \in X$, and take a +ve number r. Then in NLS X, the set $\{x \in X : \| x - x_0 \| < r\}$ is called an open ball denoted by $B_r(x_0)$ centred at x_0 having radius = r. Similarly, we have a closed ball $\overline{B}_r(x_0) = \{x \in x : \| x - x_0 \| \leq r\}$; and in agreement with usual open sphere we encounter in Co-ordinate Geometry we have a sphere $S_r(x_0) = \{x \in X : \| x - x_0 \| = r$ centred at x_0 with radius = r.

Definition 2.3.2. A *NLS* (X, || ||) is said to be a Banach space if it is a complete metric space with metric induced from the norm function || || on X.

Example 2.3.1. The space $C[a,b]$ of all real-valued continuous functions over closed interval $[a,b]$ is a Banach space with supnorm $\| f \| = \sup_{a \leq t \leq b} | f(t) |; f \in C[a,b]$.

Solution : It is routine exercise to see that $C[a,b]$ is a real linear space in respect of usual addition and scalar multiplication of continuous functions.

Now put $\| f \| = \sup_{a \leq t \leq b} | f(t) |$ for $f \in C[a,b]$ wherein we recall that $|f|$ is also continuous function over closed interval $[a,b]$ with a finite sup value $= \| f \| \geq 0$. Also $\| f \| = 0$ if and only f equals to the zero function. So (N.1) axiom is satisfied; For (N.2) take α any scalar (real), then we have for $f \in C[a,b]$,

$$\| \alpha f \| = \sup_{a \leq t \leq b} \| (\alpha f)(t) | = \sup_{a \leq t \leq b} \| \alpha\, f(t) | = | \alpha | \sup_{a \leq t \leq b} | f(t) | = | \alpha | \| f \|.$$

Also, if $f, g \in C[a,b]$ we have $\| f + g \| = \sup_{a \leq t \leq b} | (f + g)(t) |$

$$= \sup_{a \leq t \leq b} | f(t) + g(t) | \leq \sup_{a \leq t \leq b} | f(t) | + \sup_{a \leq t \leq b} | g(t) | = \| f \| + \| g \|.$$

Thus $C[a,b]$ is a NLS; Now take $\{f_n\}$ as a Cauchy sequence in $C[a,b]$; So $\| f_n - f_m \| \to 0$ as, $n, m \to \infty$. Give a $\varepsilon > 0$, we find an index N satisfying

$$\| f_n - f_m \| < \varepsilon \text{ whenever } n, m \geq N.$$

That is, $\sup_{a \le t \le b} | f_n(t) - f_m(t) | < \varepsilon$ ....

Thus for $a \le t \le b$, we have $| f_n(t) - f_m(t) | \le \sup_{a \le t \le b} | f_n(t) - f_m(t) | < \varepsilon$ whenever $n, m \ge N$. Above inequality shows that the sequence $\{f_n\}$ of continuous functions over the closed interval $[a,b]$ converges uniformly to a function say f over $[a,b]$ and also f becomes a continuous function over $[a,b]$. So $f \in C[a,b]$. Taking $m \to \infty$ in (2.3.1) we find

$$| f_n(t) - f(t) | \le \varepsilon \text{ whenever } n \ge N \text{ and for all } t \text{ in } a \le t \le b.$$

This gives $\sup_{a \le t \le b} | f_n(t) - f(t) | \le \varepsilon$ whenever $n \ge N$

or, $\| f_n - f \| \le \varepsilon$ for $n \ge N$

That means, $\lim_{n \to \infty} f_n = f \in C[a,b]$. Thus $C[a,b]$ is a Banach space.

Theorem 2.3.1. Let X be a *NLS* with norm $\| \ \|$. Then

(a) $\big| \| x \| - \| y \| \big| \le \| x - y \|$ for any two members $x, y \in X$.

(b) $\| \ \| : X \to$ Reals is a continuous function.

Proof : (a) We write $\| x \| = \| x - y + y \| \le \| x - y \| + \| y \|$

or, $\| x \| - \| y \| \le \| x - y \|$ .... (2.3.1)

Interchanging x and y we have

$\| y \| - \| x \| = \| y - x \| = \| x - y \|$ .... (2.3.2)

From (2.3.1) and (2.3.2) we write

$$\pm (\| x \| - \| y \|) \le \| x - y \|$$

or, $\big| \| x \| - \| y \| \big| \le \| x - y \|$

(b) Let $\{x_n)$ be a sequence of elements in X converge to x_0.

So $\| x_n - x_0 \| \to 0$ as $x \to \infty$. By (a) we have

$$\big| \| x_n \| - \| x_0 \| \big| \le \| x_n - x_0 \| \to 0 \text{ as } n \to \infty.$$

That means, $\lim_{n \to \infty} \| x_n \| = \| x_0 \|$. Hence norm function $\| \ \|$ is continuous at x_0; As x_0 may be taken as any point in X, (b) follows.

Remark : If $\lim_{n \to \infty} x_n = x_0$ and $\lim_{n \to \infty} y_n = y_0$ in *NLS X*, then

(a) $\lim_{n\to\infty}(x_n \pm y_n) = x_0 \neq y_0$

(b) $\lim_{n\to\infty}(\lambda x_n) = \lambda x_0$ for any scalar λ.

Definition 2.3.3. Two norms $\|\ \|_1$ and $\|\ \|_2$ in a linear space X are said to be equivalent if there two +ve constants a and b such that

$$a\|x\|_2 \leq \|x\|_1 \leq b\|x\|_2 \text{ for all } z \in X.$$

Example 2.3.2. Consider $NLS = R^2$ (Euclidean 2-space) with two norms $\|\ \|_1$ and $\|\ \|_2$ defined by $\|x, y\|_1 = \sqrt{x^2 + y^2}$ and $\|x, y\|_2 = \max(|x|, |y|)$ for $(x, y) \in R^2$. Show that two norms are equivalent.

Solution : We have for $(x, y) \in R^2$, $|x|^2 \leq |x|^2 + |y|^2$ and $|y|^2 \leq |x|^2 + |y|^2$

Thus $\|(x, y)\|_2 = \max(|x|, |y|) \leq \sqrt{|x|^2 + |y|^2} = \sqrt{x^2 + y^2} = \|(x, y)\|_1$

or, $\|(x, y)\|_2 \leq \|(x, y)\|_1$ (2.3.1)

Again $\|(x, y)_1{}^2 = x^2 + y^2 = |x|^2 + |y|^2 \leq 2\{\max(|x|, |y|)\}^2 = 2\|(x, y)\|_2$

or, $\|(x, y)_1 \leq \sqrt{2}\|(x, y)\|_2$ (2.3.2)

Combining (2.3.1) and (2.3.2) we produce

$$\|(x, y)\|_2 \leq \|(x, y)\|_1 \leq \sqrt{2}\|(x, y)\|_2$$

Therefore two norms as given are equivalent in $NLS = R^2$.

Explanation : If two norms $\|\ \|_1$ and $\|\ \|_2$ are equivalent in a NLS X, then identify function : $(X, \|\ \|_1) \to (X, \|\ \|_2)$ is a homeomorphism. (In fact, it is a linear homeomorphism).

§ 2.4 QUOTIENT SPACE :

Let $(X, \|\ \|)$ be a NLS and F be a linear sub-space of X.

If $x \in X$, let $x + F = \{x + y : y \in F\}$.

These subsets $x + F$ as $x \in X$ are cosets of F in X.

Put $X/F = \{x + F : x \in X\}$.

One observes that $F=\underline{0}+F$, $x_1+F=x_2+F$ if and only if $x_1-x_2\in F$, and as a result, for each pair $x_1,x_2\in X$, either $(x_1+F)\cap(x_2+F)=\Phi$

or, $x_1+F=x_2+F$

Further, if $x_1,x_2,y_1,y_2\in X$, and $(x_1-x_2)\in F$, $(y_1-y_2)\in F$, then

$(x_1+y_1)-(x_2+y_2)\in F$, and for any scalar α $(\alpha x_1-\alpha x_2)\in F$ because F is Linear sub-space.

We define two operations in $X\backslash F$ by the following rule :-

(i) $(X\backslash F\times(X\backslash F)\to(X|F)$

where $(x+F,y+F)\to(x+F)+(y+F)=(x+y)+F$

and (ii) $R(¢)\times(X\backslash F)\to(X|F)$

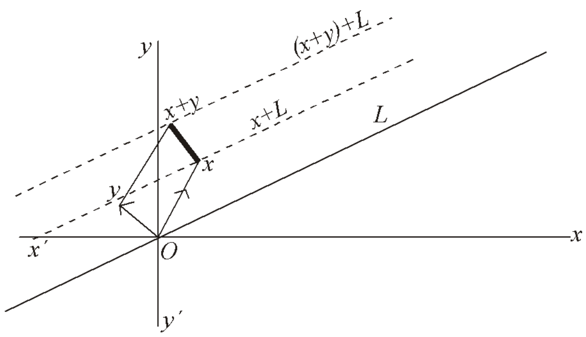
where $(\alpha,x+F)\to\alpha(x+F)=\alpha x+F$

for all $x,y\in X$ and α any scalar.

It is now a routine exercise to verify that $(X\backslash F)$ is a linear space in respect of above 'addition' and 'scalar multiplication'. Note that zero vector of this Linear space $(X|F)$ equals to F.

Definition 2.4.1. The linear space $X\backslash L$ where L is a linear subspace of *NLS X* is called the quotient space (or quotient space of X modulo L).

Example 2.4.1. Geometrically describe the quotient space R^2/L where R^2 = the Euclidean 2-space and L is the sub-space represented by a line through origin $(0,0)\in R^2$.



Solution : Given a sub-space L as represented by a line through $(0,0)\in R^2$, $\underset{\sim}{x}$ is any position of R^2, then $\underset{\sim}{x}+L$ geometrically represents a straight line through $\underset{\sim}{x}$ parallel to the line represented by L; that is say that $\underset{\sim}{x}+L$ is a translate of L through

x. Further if y is any other position of R^2, then by Law of parallelogram we obtain the position $x + y$ and here $(x+L)+(y+L)=(x+y)+L$ is re-presented by the straight line through $x + y$ and it is parallel to L; that is—it is the translate of L through $(x + y)$ in R^2.

Example 2.4.2. Obtain the quotient space $C[0,1]/L$ where $C[0,1]$ is the linear space of all real valued continuous functions over the closed interval $[0,1]$ and L consists of those members $f \in C[0,1]$ with $f(1) = 0$, *i.e.* vanishing at $t = 1$.

Solution : If $f, g \in L$, then $f(1) = g(1) = 0$; Now $(f+g)(1) = f(1)+g(1) = 0$; So $f+g \in L$ (note that sum of two continuous functions over $[0,1]$ is again a continuous functions over $[0,1]$), and for any scalar α we have $\alpha f \in L$ when $f \in L$. Therefore L is a sub-space of $C[0,1]$.

Let us look at members of $C[0,1]\backslash L$. Take $f \in C[0,1]$ where $f(1) = a$ (say). Then for any other member $g \in C[0,1]$ sharing the value a at $t = 1$, *i.e.* $g(1) = a$; we note that $(g-f) \in C[0,1]$ such that $(g-f)(1) \in g(1) - f(1) = a - a = 0$; showing that $(g-f) \in L$ *i.e.* $g \in f+L$. So these members g plus f all belong to $f + L$.

Now if $h \in C[0,1]$ with $h \notin (f+L)$ (2.4.1)

So, $h - f \notin L$

i.e. h and f differ at $t = 1$.

i.e. $h(1) \neq f(1) = a$

We similarly construct a member $(h + L)$ of $C\backslash L$, where

$$(h+L) \cap (f+L) = \phi \qquad (2.4.2$$

or else, we find a member ϕ in both implying

$$\varphi - h \in L \quad \text{and} \quad \varphi - f \in L$$

therefore $\varphi(1) - h(1) = 0$ and $\varphi(1) - f(1) = 0$

i.e. $\varphi(1) = h(1)$ and $\varphi(1) = f(1)$

i.e. $h(1) = f(1)$

that means $h \in (f+L)$, which is not the case by (2.4.1).

Theorem 2.4.1. Let L be a closed linear sub-space of *NLS X*, and let $\| x+L \| = Inf\{\| x+y \| : y \in L\}$, for all $x \in X$, then above is a norm function on the quotient space $(X \setminus L)$. Further if X is Banach space, so will be $(X \setminus L)$.

Proof : For any member $x + L$ of $X \backslash L$, from definition we have

$$\| x+L \| \geq 0 \text{ for any } x \in X.$$

Now assume that $\| x+L \| = 0$ for some $x \in X$.

i.e. $Inf\{\| x+y \| : y \in L\} = 0$

As $y \in L$ if and only if $-y \in L$, we have

$$Inf\{\| x-y \| : y \in L\} = 0.$$

Since L is closed, $x \in L$ (distance of x from L is zero);

That means $x+F = F =$ the zero vector of the quotient space X/L.

For verification (N.2) take α any non-zero scalar. Then

$$\| \alpha(x+L) \| = \| \alpha x + L \|$$

$$= Inf\{\| \alpha x + y \| : y \in L\}$$

$$= Inf\{\| \alpha(x + \tfrac{y}{\alpha}) \| : y \in L\}$$

$$= |\alpha| \, Inf\{\| x + (\tfrac{1}{\alpha}) y \| : y \in L\}$$

$$= |\alpha| \| x+L \| \text{ , because } L \text{ is a linear sub-space of } X.$$

For triangle inequality (N.3) take $x, y \in L$

Then $\| (x+t)+(y+L) \| = \| (x+y)+L \|$ (L is a linear sub-space).

$$= Inf\{\| x+y+u \| : u \in L\}$$

$$= Inf\{\| x+y+\tfrac{u}{2}+\tfrac{u}{2} \| : u \in L\}$$

$$\leq Inf\{\| x+\tfrac{u}{2} \| + \| y+\tfrac{u}{2} \| : u \in L\}$$

$$\leq Inf\{\| x+\tfrac{u}{2} \| : u \in L\} + Inf \| y+\tfrac{u}{2} \| : u \in L\}$$

$$= Inf\{\| x+h \| : u \in L\} + Inf \| y+K \| : K \in L\}; \text{ } L \text{ is a sub-space.}$$

$$= \| x+L \| + \| y+L \|$$

Thus quotient space $X \backslash L$ is a *NLS.*

Now suppose X is a Banach space. We show that the quotient space X/L is so. Let $\{x_n + L\}$ be a Cauchy sequence in (X/L). So corresponding to each +ve integer k we find an index N_k such that

$$\| x_n - x_m + L \| < \frac{1}{2^k}, \text{ whenever } m, n \geq N_k \qquad (2.4.1)$$

We define by Induction a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\| x_{n_k} - x_{n_{k+1}} + L \| < \frac{1}{2^k}$$

Take $n_1 = N_1$, and suppose $n_2, n_3, ..., n_k$ have been so defined that $n_1 < n_2 < ... < n_k$ and $N_j \leq n_j$ (j = 1,2,....,k).

Let $n_{k+1} = \max\{N_{k+1}, n_k + 1\}$. This enables one to obtain an increasing sequence $\{n_k\}$ and (*) follows from (2.4.1)

Put $y_k = x_{n_k}$. Then by induction we define a sequence $\{z_k\}$ in L such that $z_k \in (y_k + L)$ and $\| z_k - z_{k+1} \| < \frac{1}{2^{k+1}}$, k = 1,2, ...

Choose $z_1 \in (y_1 + L)$, suppose $z_2, ..., z_k$ have been so chosen to satisfy above condition. Then $y_k + L = z_k + L$ and by (2.4.1) we have $\| z_k - y_{k+1} + F \| < \frac{1}{2^k}$. By definition of norm in $(X \backslash L)$

we find $z_{k+1} \in (y_{k+1} + L)$ such that $\| z_k + z_{k+1} \| \leq z_k - y_{k+1} + L \| + \frac{1}{2^k}$.

Then $\| z_k + z_{k+1} \| < \frac{1}{2^{k+1}}$ as wanted.

That means $\sum_{k=1}^{\infty} \| z_k - z_{k+1} \|$ is convergent, and hence $\sum_{k=1}^{\infty} (z_k - z_{k+1})$ is convergent.

But $\sum_{k=1}^{\infty} (z_k - z_{k+1}) = (z_1 - z_2) + (z_2 - z_3) + ... + (z_m - z_{m+1}) = z_1 - z_{m+1}$.

So, $\{z_m\}$ is convergent; Put $\lim_{k\to\infty} z_k = z$; since $z_k \in (y_k + L)$ we have $\|(z+L)-(y_k+L)\| = \|z - y_k + L\| \le \|z - z_k\|$.

That means $\lim_{k\to\infty}\{y_k + L\} = z + L$. Thus given Cauchy sequence $\{x_n + L\}$ has a convergent subsequence $\{x_{n_k} + F\}$.

Hence $\{x_n + L\}$ is convergent in $(X\backslash L)$. This proves that $(X\backslash L)$ is a Banach space.

§ 2.5 CONVEX SETS IN *NLS* :

Let $(X, \|\ \|)$ be a *NLS*, and C be a non-empty subset of X.

Definition 2.5.1. C is said to be a convex set if for any real scalar α in $0 \le \alpha \le 1$, and any two members $x_1, x_2 \in C$ we have $\alpha x_1 + (1-\alpha)x_2$ is a member of C.

Or, equivalently, for any two reals α, β with $0 \le \alpha, \beta \le 1$ $\alpha + \beta = 1$, $(\alpha x_1 + \beta x_2) \in C$.

Or, equivalently, the segment consisting of members $tx_1 + (1-t)x_2$ $(0 \le t \le 1)$ is a part of C.

For example, in an Euclidean space like R^n, cubes, ball, sub-spaces are all examples of convex sets in R^n.

Theorem 2.5.1. Intersection of any number of convex sets in a *NLS* is a convex set, but their union may not be so,

Proof : Suppose $\{C_\alpha\}_{\alpha\in\Delta}$ be a family of convex set in *NLS* $(X, \|\ \|)$ and put $C = \bigcap_{\alpha\in\Delta} C_\alpha$; Let $C \ne \phi$ and let $x, y \in C$ take $0 \le \alpha \le 1$. Now $x, y \in \bigcap_{x\in\Delta} C_\alpha$, so for every α, x and y are members of C_α which is convex, thus, $(\alpha x + (1-\alpha)y \in C_\alpha$. Therefore $\alpha x + (1-\alpha)y$ is a member of every C_α and hence is a member of $\bigcap_{x\in\Delta} C_\alpha = C$. Thus C is shown to be a convex set in X.

Union of two convex sets may not be a convex set. Every triangular region in Euclidean plane is a convex set but the figure ⧖ as a union of two such convex sets fails to be a convex set.

Theorem 2.5.2. A subset C in a *NLS* is convex if and only if $sC + tC = (s+t)C$ for all +ve scalars s and t.

Proof : For all scalars s and t we have

$$(s+t)C \subset sC + tC \qquad \text{(2.5.1)}$$

If C is convex and s, t are +ve scalars we have

$$\frac{s}{s+t}C + \frac{t}{s+t}C \subset C$$

Or $\quad sC + tC \subset (s+t)C \qquad \text{(2.5.2)}$

Combining (2.5.1) and (2.5.2) we have

$$sC + tC = (s+t)C$$

Conversely, suppose $(s+t)C = sC + tC$ holds for all +ve scalars; If $0 \le \alpha \le 1$, take $s = \alpha$ and $t = 1-\alpha$ and then we find $\alpha C + (1-\alpha)C \subset C$. So C is convex.

Theorem 2.5.3. A ball (open or closed) of a *NLS* is a convex set.

Proof : $\bar{B}(x_0, r)$ be a closed ball in a *NLS* $(X, \|\ \|)$.

Let $x, y \in \bar{B}(x_0, r)$; So $\|x - x_0\| \le r$ and $\|y - x_0\| \le r$. If $0 \le t \le 1$, and $u = tx + (1-t)y$, we have

$$\|u - x_0\| = \|tx + (1-t)y - (tx_0 + (1-t)x_0)\| = \|t(x - x_0) + (1-t)(y - y_0)\|$$

$$\le t\|x - x_0\| + (1-t)\|y - y_0\| \le tr + (1-t)r = r.$$

That shows $u \in \bar{B}(x_0, r)$. So, $\bar{B}(x_0, r)$ is shown to be convex. The proof for an open ball shall be similar.

Example 2.5.1. If $(X, \|\ \|)$ is a Banach space and L is a closed sub-space of X, show that L is a Banach space.

Solution : If L is a closed sub-space of X, then L becomes a closed set of a complete metric space X, the metric being induced from the norm $\|\ \|$. And we know that every closed sub-space of a complete metric space is a complete metric sub-space and hence here L is a Banach space. (as a sub-space of X).

§ 2.6 BOUNDED LINEAR OPERATORS OVER A *NLS* $(X, \|\ \|)$:

Let $(X, \|\ \|)$ and $(Y, \|\ \|)$ be two *NLS* with same scalar field. (Here, same notation $\|\ \|$ has been used for norm function; it is to be noted that norm functions in X and Y are, in general, different).

Definition 2.6.1. A function (or mapping or transformation) (function, map, mapping, transformation are synonyms of the same mathematical object) $T : X \to Y$ is called a linear operator if (1) $T(x_1 + x_2) = T(x_1) + T(x_2)$ for any two members x_1 and x_2 in X, and

(2) $T(\alpha x_1) = \alpha T(x_1)$ for any scalar α and for any member $x_1 \in X$.

Explanation : For a linear operator $T : X \to Y$ condition (1) in Definition 2.6.1 is termed as linearity condition which says Image of the sum is equal to sum of the images. Condition (2) is known as that homogeneity. For example, if $X = Y = R =$ the space of reals with usual norm (Euclidean norm) and $T : R \to R$ is given by $T(x) = \alpha x$ where $x \in R$ and α is a fixed real (zero or non-zero), we verify that T is a linear operator; and we shall presently see that any linear operator : $R \to R$ shall be of the form $T(x) = \alpha x$ for some fixed scalar α for all $x \in R$.

Definition 2.6.2. The operator $T : X \to Y$ defined by $T(x) = \underline{0}$ in Y. For all X, is called **the zero operator, denoted by 0**.

Remark : (a) The zero operator : $X \to Y$ is a Linear operator.

(b) The identity operator, $I : X \to X$ where $I(x) = x$ for all $x \in X$ is a linear operator.

Theorem 2.6.1. Let $T : X \to Y$ be a linear operator. If T is continuous at one point of X, then T is continuous at every other point of X.

Proof : Suppose T is continuous at $x_0 \in X$; so given $\varepsilon > 0$, there is a +ve δ such that $\| T(x) - T(x_0) \| < \varepsilon$ whenever $\| (x) - (x_0) \| < \delta$. Suppose $x_1 (\neq x_0)$ be another point of X. Then if $\| x - x_1 \| < \delta$, we write $\| x - x_1 \| = \| x_0 - (x - x_1 + x_0) \|$.

Thus $\| (x - x_1 + x_0) \| < \delta$ shall give by virtue of continuity of T at x_0,

$$\| T(x - x_1 + x_0) - T(x_0) \| < \varepsilon$$

or, $\| T(x) - T(x_1) + T(x_0) - T(x_0) \| < \varepsilon$ because T is linear.

or, $\| T(x) - T(x_1) \| < \varepsilon$. Therefore T is continuous at $x = x_1$.

Corollary : A linear operator over a *NLS* X is continuous **either everywhere or nowhere in X.**

Definition 2.6.3. A linear operator $T : X \to Y$ is called bounded if there is a +ve constant M such that

$$\| T(x) \| \le M \| x \| \text{ for all } x \in X.$$

or equivalently $\dfrac{\| T(x) \|}{\| x \|} \le M$ for all non-zero numbers $x \in X$.

Theorem 2.6.2. Let $T : X \to Y$ be a linear operator. Then T is continuous if and only if T is bounded.

Proof : Let $T : X \to Y$ be a continuous linear operator; if possible let T be not bounded. So for every +ve integer n we find a member $x_n \in X$ such that

$$\| T(x_n) \| > n \| x_n \| \text{ .... } (2.6.1)$$

Now x_n is non-zero vector in X, put $u_n = \dfrac{x_n}{n \| x_n \|}$,

clearly $\| x_n \| = \dfrac{1}{n} \cdot \dfrac{1}{\| x_n \|} = \dfrac{1}{n} \to 0$ as $n \to \infty$. So we see $\lim_{n\to\infty} u_n = \underline{0}$ in X; By continuity of T we have $\lim_{n\to\infty} T(u_n) = T(\underline{0}) = \underline{0}$ in Y. ($T(\underline{0}) = \underline{0}$, because T is linear);

Therefore we have $\| T(u_n) \| \to 0$ as $n \to \infty$ (*)

On the other hand, $\| T(u_n) \| = \left\| T\left(\dfrac{x_n}{n \| x_n \|} \right) \right\|$,

$$= \left\| \frac{1}{n \| x_n \|} T(x_n) \right\|, \text{ because } T \text{ is linear}$$

$$= \frac{1}{n} \cdot \frac{1}{\| x_n \|} \| T(x_n) \| > 1 \text{ by (2.6.1)}$$

Now $\| T(u_n) \| > 1$ and (*) are contradictory.

So, we have shown that $T : X \to Y$ is bounded.

Conversely, suppose linear operator $T : X \to Y$ is bounded. Then we find a +ve scalar such that

$$\| T(x) \| \le M \| x \|;$$

So given $\in > 0$, there is a +ve $\delta = \frac{\in}{2M}$ (here), such that

$$\| T(x) \| < \varepsilon \quad \text{whenever } \| x \| < \delta$$

i.e. $\| T(x) - T(\underline{0}) \| < \varepsilon$ whenever $\| x - \underline{0} \| < \delta$ because $T(\underline{0}) = \underline{0}$ in Y. That means, T is continuous at $x = \underline{0}$ in X, and therefore Theorem 2.6.1 says that T is continuous at every non-zero position of X. The proof is now complete.

Examples of bounded and unbounded linear operators.

Example 2.6.1. Consider a transformation T of rotation in Euclidean 2-space R^2 given by $T(x, y) \to (x', y')$ where

$$\left. \begin{aligned} x' &= x\cos\theta + y\sin\theta \\ y' &= -x\sin\theta + y\cos\theta \end{aligned} \right\} \qquad (*)$$

Now it is easy to verify that $T : R^2 \to R^2$, under (*) is a linear operator in respect which rotation takes place around origin (0,0) with axes of co-ordinates being rotated through angle θ to give new axes of co-ordinates.

In *NLS* R^2 with usual norm $\| (x, y) \| = x^2 + y^2$, we see that

$$\| T(x, y) \|^2 = \| (x', y') \| = x'^2 + y'^2 = (x\cos\theta + y\sin\theta) + (-x\sin\theta + y\cos\theta)^2$$

$$= x^2 + y^2 = \| (x, y) \|^2.$$

Thus $\| T(x, y) \| = \| (x, y) \|$; and this is true for all points (x, y) in R^2, and we conclude that T is a bounded linear operator.

Example 2.6.2. Consider the Banach space $C[0,1,]$ of all real-valued continuous functions over the closed interval [0,1] with respect to sup norm

$$\| f \| = \sup_{a \le t \le 1} | f(t) |; \; f \in C[0,1]$$

Let $K(s,t)$ be a real-valued continuous function over the square

$$\{0 \le s \le t \,;\, 0 \le t \le 1\}.$$

Now define $T : C[0,1] \to C[0,1]$ by the rule : let $T(f) = F$

where $F(s) = \int_0^1 k(s,t) f(t) dt$; as $f \in C[0,1]$.

It is a routine exercise to check that F is continuous over $[0,1]$ and T is a linear operator.

Now, $\|T(f)\| = \|F\| = \sup_{0\le s\le 1} |F(s)| = \sup_{0\le s\le 1} |\int_0^1 k(s,t)f(t)dt|$

$\le \sup_{0\le s\le 1} \int_0^1 |k(s,t)||f(t)|dt \le M\int_0^1 |f(t)|dt$ where $M = \sup_{0\le s\le 1,\, 0\le t\le 1} |k(s,t)|$;

$\le M.\sup_{0\le t\le 1} |f(t)|\int_0^1 dt = M.\|f\|$. This is true for every member $f \in C[0,1]$.

Therefore, T is shown to be bounded.

Example 2.6.3. Let $C^{(1)}[0,1,]$ denote the class of real-valued continuous functions that are continuously differentiable over $[0,1]$. Then $C^{(1)}[0,1]$ is a sub-space of $C[0,1]$ which is Banach space with sup norm. Consider the Differential operator $D: C^{(1)}[0,1] \to C[0,1]$ when $D(f) = \varphi$; $f \in C^{(1)}[0,1]$ and $\frac{d}{dt} f(t) = \varphi(t)$ in $0 \le t \le 1$. We can easily verify that D is a linear operator; presently we see that D is not bounded.

Let us take $f_n \in C^{(1)}[0,1]$ where $f_n(t) = \sin n\pi t$ in $0 \le t \le 1$. Then we have $Df_n = \varphi_n$ where $\varphi_n(t) = \frac{d}{dt}(\sin n\pi t) = n\pi \cos n\pi t$ in $0 \le t \le 1$.

Therefore, $\|f_n\| = \sup_{0\le t\le 1} |\sin n\pi t| = 1$ and

$$\|D(f_n)\| = \|\varphi_n\| = \sup_{0\le t\le 1} |n\pi \cos n\pi t| = n\pi$$

Here $\frac{\|D(f_n)\|}{\|f_n\|} = \frac{n\pi}{1} \to \infty$ as $n \to \infty$

That means D can not be bounded.

Definition 2.6.4. Let $T : X \to Y$ be a bounded (or equivalently, continuous) linear operator. Then the norm of T, denoted by $\|T\|$ is defined as

$$\|T\| = Inf\{M > 0 : \|T(x)\| \le M\|x\| \text{ for all } x \in X\}$$

(A set of +ve reals has always *Inf.* value).

Theorem 2.6.3. Let $T : X \to Y$ be a bounded linear operator. Then

(a) $\| T(x) \| \le \| T \| \| x \|$ for all $x \in X\}$

(b) $\| T \| = \sup_{\|x\| \le 1} \{\| T(x) \|\}$

(c) $\| T \| = \sup_{\|x\| = 1} \{\| T(x) \|\}$

(d) $\| T \| = \sup_{x \ne \underline{0}} \left\{ \frac{\| T(x) \|}{\| x \|} \right\}$

Proof : (a) From definition of operator norm we see that for any +ve ε we have $\| T(x) \| \le (\| T \| + \varepsilon) \| x \|$ for all $x \in X$.

Taking $\varepsilon \to 0_+$ we have $\| T(x) \| \le \| T \| \| x \|$

(b) If $\| x \| \le 1, x \in X$, we have $\| T(x) \| \le \| T \| \| x \| \le \| T \|$

Therefore $\sup_{\|x\| \le 1} \| T(x) \| \le \| T \|$ (1)

From Definition of operator norm $\| T \|$, given any +ve ε, we find $x_\varepsilon \in X$ such that $\| T(x_\varepsilon) \| > (\| T \| - \varepsilon) \| x \|$.

Take $u_\varepsilon = \frac{x_\varepsilon}{\| x_\varepsilon \|}$ we see $\| u_\varepsilon \| = 1$ such that

$$\| T(u_\varepsilon) \| = \frac{1}{\| x_\varepsilon \|} \| T(x_\varepsilon) \| > \frac{1}{\| x_\varepsilon \|} .(\| T \| - \varepsilon) \| x_\varepsilon \| = \| T \| - \varepsilon$$

As $\| u_\varepsilon \| = 1$, this gives $\sup_{\|x\| \le 1} \| T(x) \| \ge \| T(u_\varepsilon) \| > \| T \| - \varepsilon$. As $\varepsilon > 0$ is arbitrary we produce $\sup_{\|x\| \le 1} \| T(x) \| \ge \| T \|$ (2)

From (1) and (2) we have (b), namely, $\sup_{\|x\| \le 1} \| T(x) \| = \| T \|$

(c) the proof shall be like that of (b).

(d) we have $\| T(x) \| \le \| T \| \| x \|$ for all $x \in X$.

So, $\frac{\|T(x)\|}{\|x\|} \leq \|T\|$ for $x \in X$ with $x \neq 0$.

Since r.h.s does not depend on non-zero $x \in X$, we have

$$\sup_{x \neq \underset{\sim}{0}} \frac{\|T(x)\|}{\|x\|} \leq \|T\| \tag{3}$$

Again given a +ve ε $(0 < \varepsilon < \|T\|)$ we find a member $x_\varepsilon \in X$ such that

$\|T(x_\varepsilon)\| > (\|T\| - \varepsilon)\|x_\varepsilon\|$; clearly $x_\varepsilon \neq \underset{\sim}{0}$.

Thus $\frac{\|T(x_\varepsilon)\|}{\|x_\varepsilon\|} > \|T\| - \varepsilon$

Therefore $\sup_{x \neq \underset{\sim}{0}} \frac{\|T(x)\|}{\|x\|} \geq \frac{\|T(x_\varepsilon)\|}{\|x_\varepsilon\|} > \|T\| - \varepsilon$

Now taking $\varepsilon \to 0_+$ we find

$$\sup_{x \neq \underset{\sim}{0}} \frac{\|T(x)\|}{\|x\|} \geq \|T\| \tag{4}$$

Combining (3) and (4) we have $\sup_{x \neq \underset{\sim}{0}} \frac{\|T(x)\|}{\|x\|} = \|T\|$.

EXERCISE A

Short answer type questions :

1. In a linear space X if $x \in X$ show that $-(-x) = x$.
2. If a finite set of vectors in a linear space contains the zero vector show that it is a linearly dependent set.
3. In Euclidean 2-space R^2 describe geometricaly open ball centred at (0,0) with radius = 1 in respect of (a) $\|x\|_1 = \sqrt{x_1^2 + x_2^2}$ (b) $\|x\|_2 = |x_1| + |x_2|$ and (c) $\|x\|_3 = \max\{|x_1|, |x_2|\}$ where $x = (x_1, x_2) \in R^2$.
4. Obtain a condition such that function suit and $\sin \lambda t$ are linearly independent in the space $C[0,2\pi]$.
5. Construct a basis of Euclidean 3-space R^3 containing (1,0,0) and (1,1,0).

EXERCISE B

Broad asnwer type questions

1. If $C[a,b]$ is the linear space of all real-valued continuous functions over the closed interval $[a,b]$, show that $C[a,b]$ is a Normed Linear space with respect to $\|f\| = \int_a^b |f|\,dt$, $f \in C[a,b]$. Examine if $C[a,b]$ is a Banach space with this norm.
2. In a *NLS X*. verify that for a fixed member $a \in X$, the function $f : X \to X$ given by $f(x) = x + a$; $x \in X$ is a homeomorphism. Hence deduce that translate of an open set in X is an open set.
3. Examine if the sub-space $\rho[0,1]$ of all real polynomials over the closed interval $[0,1]$ is a closed sub-space of the Banach space $C[0,1]$ with sup norm.
4. Prove that in a *NLS* the closure of the open unit ball is the closed unit ball.
5. Let $(X, \|\ \|)$ and $(Y, \|\ \|)$ be two *NLS* over the same scalars and $T : X \to Y$ be a linear operator that sends a convergent sequence in X to a bounded sequence in Y. Prove that T is a bounded linear operator.
6. Let T : $C[0,1] \to$ itself, where $C[0,1]$ is the Banach space of all real-valued continuous functions over the closed unit interval with sup norm such that $T(x) = y$ where

   $$y(t) = \int_0^t x(u)\,du; \quad x \in C[0,1] \text{ and } 0 \le t \le 1$$

   Find the range of T, and obtain $T^{-1} : (\text{range}\, T) \to C[0,1]$.

   Examine if T^{-1} is linear and bounded.