UNIT 3

(***Contents*** : Every Finite Dimensional *NLS* is a Banach space, Equivalent norms, Riesz Lemma, Finite Dimensionality of *NLS* by compact unit ball, Linear operators over finite Dimensional *NLS* and matrix representation; Isomorphism, Boundedness of linear operators over finit Dimensional *NLS*, space *Bd𝓛(X,Y)* of bounded linear operators, and its completeness).

§ 3.1 FINITE DIMENSIONAL *NLS*

Theorem 3.1.1. Every finite dimensional *NLS* is a Banach space. To prove this Theorem we need a Lemma.

Lemma 3.1.1. Let $(x_1, x_2, ..., x_n)$ be a set of linearly independent vectors in a *NLS* $(X, \|\ \|)$; then there is a +ve β such that

$\|\alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_n x_n\| \geq \beta(|\alpha_1| + |\alpha_2| + ... + |\alpha_n|)$ for every set of scalars $\alpha_1, \alpha_2 ..., \alpha_n$.

Proof : Put $S = \sum_{i=1}^{n} |\alpha_i|$. Without loss of generality we take $S > 0$.

Then above inequality is changed into

$$\|\beta_1 x_1 + \beta_2 x_2 + ... + \beta_n x_n\| \geq \beta, \text{ where } \beta_i = \frac{d_i}{S} \qquad (*)$$

and $\sum_{i=1}^{n} |\beta_i| = 1$.

If suffices to establish (*) for any set of scalars $\beta_1, \beta_2, ..., \beta_n$ with $\sum_{i=1}^{n} |\beta_i| = 1$. We apply method of contradiction. Suppose there is a sequence $\{y_m\}$ with

$y_m = \beta_1^{(m)} x_1 + \beta_2^{(m)} x_2 + ... + \beta_n^{(m)} x_n$; and $\sum_{i=1}^{n} |\beta_i^{(m)}| = 1$ for m = 1, 2, ...

such that $\|y_m\| \to 0$ as $m \to \infty$

Now $|\beta_i^{(m)}| \leq \sum_{i=1}^{n} |\beta_i^{(m)}| = 1$

Hence for a fixed i the sequence $\{\beta_i^{(m)}\} = \{\beta_i^{(1)}, \beta_i^{(2)}, ...\}$ is bounded. So Bolzano-Weirstrass Theorem says that $\{\beta_i^{(m)}\}$ has a sub-sequence that converges to (say) β_i.

Let $\{y_{1,m}\}$ denote the corresponding subsequence of $\{y_m\}$. By the same argument $\{y_{1,m}\}$ shall give a sub-sequence, say $\{y_{2,m}\}$ for which the corresponding subsequence of scalars $\{\beta_2^{(m)}\}$ converges to β_2 (say). We continue this process. At nth stage we produce a subsequence $\{y_{n,m}\} = \{y_{n,1}, y_{n,2}, ...\}$ of $\{y_m\}$ whose term

$$y_{n,m} = \sum_{i=1}^{n} \delta_i^{(m)} x_i\,, \quad \sum_{i=1}^{n} |\delta_i^{(m)}| = 1$$

such that $\lim_{m\to\infty} \delta_i^{(m)} = \beta_i$, Hence we see

$\lim_{m\to\infty} y_{n,m} = \sum_{i=1}^{n} \beta_i x_i = y$ (say) when $\sum_{i=1}^{n} |\beta_i| = 1$. That means all β_i's are not zero. Since $x_1, x_2, ..., x_n$ are linearly independent it follows that $y \neq \underset{\sim}{0}$.

Now $\lim_{m\to\infty} y_{n,m} = y$ gives

$$\lim_{m\to\infty} \| y_{n,m} \| = \| y \|.$$

Since $\{y_{n,m}\}$ is a sub-sequence of $\{y_m\}$ and $\| y_m \| \to 0$ as $m \to \infty$, So $\| y_{n,m} \| \to 0$ as $m \to \infty$ and so $\| y \| = 0$ giving $y = \underset{\sim}{0}$, a contradiction. Therefore Lemma is proved.

Proof of Theorem 3.1.1. Suppose $\{y_m\}$ be a Cauchy sequence in a finite dimensional *NLS* (X, $\|\ \|$). Let Dim(X) = n, and (e_1, e_2, ..., e_n) forms a basis in X. So each y_m has a unique representation.

$$y_m = \alpha_1^{(m)} e_1 + \alpha_2^{(m)} e_2 + ... + \alpha_n^{(m)} e_n$$

Give a +ve ε, as $\{y_m\}$ is Cauchy, we find an index N such that

$$\| y_m - y_r \| < \varepsilon \text{ for } m, r \geq N.$$

Now $\varepsilon > \| y_m - y_r \| = \| \sum_{i=1}^{n} (\alpha_i^{(m)} - \alpha_i^{(r)}) e_i \|$

$$\geq \beta \sum_{i=1}^{n} | \alpha_i^{(m)} - \alpha_i^{(r)} | \text{ by Lemma 3.1.1}$$

whenever $m, r > N$. Therefore

$$| \alpha_i^{(m)} - \alpha_i^{(r)} | \leq \sum_{i=1}^{n} | \alpha_i^{(m)} - \alpha_i^{(r)} | < \frac{\varepsilon}{\beta} \text{ for } m, r > N$$

Therefore, each of the n sequences

$\{\alpha_i^{(m)}\}$ $(i = 1, 2, ..., n)$ becomes a Cauchy sequence of scalars (reals/complex), and by Cauchy's General Principle of convergence becomes a convergent sequence with, say,

$$\lim_{m \to \infty} \alpha_i^{(m)} = \alpha_i^{(0)} \text{ (say)}, i = 1,2, ..., n.$$

Put $y = \alpha_1^{(0)} e_1 + \alpha_2^{(0)} e_2 + ... + \alpha_n^{(0)} e_n$; so $y \in X$.

Further, $\lim_{m \to \infty} \alpha_i^{(m)} = \alpha_i^{(0)}$ for $i = 1,2, ..., n$ gives,

$$\| y_m - y \| = \| \sum_{i=1}^{n} (\alpha_i^{(m)} - \alpha_i^{(0)}) e_i \| \leq \sum_{i=1}^{n} | \alpha_i^{(m)} - \alpha_i^{(0)} | \| e_i \| \to 0 \text{ as } m \to \infty.$$

i.e. $\lim_{m \to \infty} y_m = y \in X$. So given Cauchy sequence $\{y_m\}$ in X is convergent in X; and $(X, \| \ \|)$ is Banach space.

Theorem 3.2.1. Any two norms in a finite dimensional *NLS X* are equivalent.

Proof : Let Dim $(X) = n$ and $(e_1, e_2, ..., e_n)$ form a basis for X. If $x \in X$, we write $x = \alpha_1 e_1 + \alpha_2 e_2 + ... + \alpha_n e_n$ uniquely.

Applying Lemma 3.1.1 we find a +ve β such that

$$\| x \|_1 \geq \beta(| \alpha_1 | + | \alpha_2 | + ... + | \alpha_n |)$$

If $\mu = \max_{1 \leq i \leq n} \| e_i \|_2$; Then we have

$$\| x \|_2 \leq \sum_{i=1}^{n} | \alpha_i | \| e_i \| \leq \mu \sum_{i=1}^{n} | \alpha_i | \leq \frac{\mu}{\beta} \| x \|_1$$

or, $\beta_{\overline{u}} \| x \|_2 \leq \| x \|_1$, the other half of desired inequality comes by interchanging norms $\| \ \|_1$ and $\| \ \|_2$. The proof is now complete.

Theorem 3.1.3. A *NLS* (X, || ||) is finite Dimensional if and only if the closed unit ball (centred at $\underset{\sim}{0}$) is compact.

To prove this theorem we need support of another result popularly known as Riesz Lemma.

Lemma 3.1.2 (Riesz Lemma). Let L ($\neq X$) be a closed sub-space of a *NLS* (X, || ||). Given a +ve $\in$ $(0 < \in < 1)$ there is a member $y \in \left(\frac{X}{L}\right)$ with $\|y\| = 1$ such that $\|y - x\| > 1 - \varepsilon$ for all $x \in L$.

Proof : Take $y_0 \in \left(\frac{X}{L}\right)$ and put $d = dist(y_0, L)$

$$= \underset{x \in L}{Inf} \|y_0 - x\|.$$

Since L is closed and y_0 is outside L, we have $d > 0$. Given a +ve ε, choose $\eta > 0$ such that

$$\frac{\eta}{d + \eta} < \varepsilon$$

So we find a member $x_0 \in L$ such that

$$d \leq \|y_0 - x_0\| < d + \eta$$

Take $y = \frac{y_0 - x_0}{\|y_0 - x_0\|} (y_0 \neq x_0)$; then $\|y\| = 1$, and we have

$y_0 = x_0 + \|y_0 - x_0\| y$. Since y_0 is outside L, we find y also outside L *i.e.* $y \in \left(\frac{X}{L}\right)$.

If $x \in L$, we have $\|y - x\| = \left\| \frac{y_0 - x_0}{\|y_0 - x_0\|} - x \right\|$

$$= \frac{1}{\|y_0 - x_0\|} \left\| y_0 - x_0 - x\|y_0 - x_0\| \right\| = \frac{1}{\|y_0 - x_0\|} \|y_0 - x'\| \text{ (say)}$$

where $x' = x_0 + \|y_0 - x_0\| x$; clearly $x' \in L$ because $x_0, x \in L$.

Therefore, $\|y - x\| > \frac{1}{d + \eta} \|y_0 - x'\| \geq \frac{d}{d + \eta} = 1 - \frac{\eta}{d + \eta} = 1 - \varepsilon$.

The proof is now complete.

Proof of Theorem 3.1.3. First suppose that closed unit ball $\hat{B}_1(\underset{\sim}{0}) = \{x \in X : \| x \| \leq 1\}$ in a *NLS* (*X*, $\| \ \|$) is compact and hence is sequentially compact. We show that $\mathrm{Dim}(X) < \infty$.

Suppose no. take $x_1 \in X$ with $\| x_1 \| = 1$ and L_1 as the sub-space spanned by $x_1(\neq \underset{\sim}{0})$. Then L_1 is a closed sub-space of *X* without being equal to *X*. So we apply Riesz Lemma (Lemma 3.1.2) when we take $\in = \frac{1}{2}$. Then we find $x_2 \in (X \setminus L_1)$ with $\| x_2 \| = 1$ and $\| x_1 - x_2 \| > \frac{1}{2}$.

Take L_2 as the sub-space spanned by x_1 and x_2. By the argument same as above we find L_2 as a proper closed sub-space of *X* and attracts Riesz Lemma. Thus there is $x_3 \in (X \setminus L_2)$ with $\| x_3 \| = 1$ and $\| x_3 - x_1 \| > \frac{1}{2}$, $\| x_3 - x_2 \| > \frac{1}{2}$.

We continue this process to obtain a sequence $\{x_n\}$ with $\| x \| = 1$ *i.e.* $x_n \in \tilde{B}_1(\underset{\sim}{0})$ such that $\| x_n - x_m \| > \frac{1}{2}$ for $n \neq m$. That means $\{x_n\}$ does not admit if any convergent subsequence : a contradiction that $\hat{B}_i(\underset{\sim}{0})$ is sequentially compact. Hence we have shown that $\mathrm{Dim}(X) < \infty$.

Conversely let (*X*, $\| \ \|$) be finite dimensional. Then it is a well known property that a subset in *X* is norm-compact if and only if that subset is bounded and closed. Here the closed unit ball $\hat{B}_i(\underset{\sim}{0})$ is bounded; and hence it must be compact. The proof is now complete.

§ 3.2 LINEAR OPERATORS OVER FINITE DIMENSIONAL SPACES :

Let R^n denote the Euclidean *n*-space. Then an $m \times n$ real matrix

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$ defines a Linear operator $T : R^n \to R^m$ where $T(\underset{\sim}{x}) = \underset{\sim}{y}$;

$\underset{\sim}{x} = (\xi_1, \xi_2, ..., \xi_n)$ and $\underset{\sim}{y} = (\eta_1, \eta_2, ..., \eta_n)$ such that

$$\sum_{j=1}^{n} a_{ij}\xi_j = \eta_i \quad i = 1, 2, ..., m.$$

Verification is an easy exercise and is left out.

Conversely, given a linear operator $T : R^n \to R^m$. We show that it is represented by an $(m \times n)$ real matrix. Let us take $(e_1, e_2, ..., e_n)$ as a basis in R^n where $e_i = \Big(\underset{i\text{th place}}{\underline{0}\,1\,\underline{0}}\Big)$, i = 1,2,..., n. And let $f_1 = \underbrace{(1, 0, 0, ...0)}_{m \text{ places}}$, $f_2 = (0, 1, 0, 0, ...0)$, $f_n = (0, 0, ..., 1)$ form the analogous basis in R^m.

Let $T(e_j) = \underset{\sim}{a}_j \in R^m$

$$= \alpha_{1j}f_1 + \alpha_{2j}f_2 + ... + \alpha_{mj}f_m \quad \text{(say)} \quad (j = 1,2, ..., n)$$

In general, if $\underset{\sim}{x} = (\xi_1, \xi_2, ...\xi_n) \in R^m$ and if $T(\underset{\sim}{x}) = \underset{\sim}{y} \in R^m$

we have $\eta_1 f_1 + \eta_2 f_2 + ... + \eta_m f_m = \underset{\sim}{y}$ and

$$\underset{\sim}{y} = T(\underset{\sim}{x}) = T\left(\sum_{j=1}^{n} \xi_j e_j\right) = \sum_{j=1}^{n} \xi_j T(e_j) = \sum_{j=1}^{n} \xi_j \underset{\sim}{a}_j$$

$$= \sum_{j=1}^{n} \xi_j \left(\sum_{i=1}^{m} \alpha_{ij} f_i\right)$$

$$= \sum_{i=1}^{m} \left(\sum_{j=1}^{n} \alpha_{ij}\xi_j\right) f_i$$

Or, $\displaystyle\sum_{i=1}^{m} \eta_i f_i = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} \alpha_{ij}\xi_j\right) f_i$ gives $\displaystyle\eta_i = \sum_{j=1}^{n} \alpha_{ij}\xi_j$; i = 1, 2, ..., m.

Therefore, T is represented by the matrix $\left((\alpha_{ji}\right)_{m \times n}$.

Remark : Given a linear operator $T : R^n \to R^m$, there is an $(m \times n)$ matrix to represent T. Entries (reals) in this matrix depend upon the choice of basis in underlying spaces. If basis changes co-efficients entering representative matrix change; However order of the matrix does not change.

Example 3.2.1. Let $\rho_3[0,1]$ denote the linear space of all real polynomials over the closed interval [0,1] with degree ≤ 3. Let $D : \rho_3[0,1] \rightarrow \rho_2[0,1]$ be the differential operator. Show that D is a linear operator and obtain a representative matrix for D.

Solution : Here $\rho_3[0,1]$ (and similarly $\rho_2[0,1]$) is a real linear space with Dim $\rho_3[0,1] = 4$ (Dim($\rho_2[0,1] = 3$). Let us take (p_0, p_1, p_2, p_3) as a basis for $\rho_3[0,1]$ where $p_0(t) = 1,\ p_1(t) = t,\ p_2(t) = t^2$ and $p_3(t) = t^3$ in $0 \leq t \leq 1$.

Then we have $D(p_0) = 0$, $D(p_1) = 1$, $D(p_2) = 2t$ and $D(p_3) = 3t^2$; and we write

$$
\begin{aligned}
0 &= 0p_0 + 0p_1 + 0p_2 \\
1 &= 1p_0 + 0p_1 + 0p_2 \\
2t &= 0p_0 + 2p_1 + 0p_2 \\
\text{and} \qquad 3t^2 &= 0p_0 + 0p_1 + 3p_2
\end{aligned}
$$

And therefore representative matrix $((a_{ij}))_{3\times 4}$ for D is given by

$$
\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}_{3\times 4}
$$

Remark : Representative matrix for linear operator changes if basis is changed.

Example 3.2.2. Let $\rho_3[0,1]$ denote the linear space of all real polynomials over the closed interval [0,1] with degree ≤ 3.

Let $T : \rho_3[0,1] \rightarrow \rho_3[0,1]$ be a linear operator given by

$T(a_0 + a_1x + a_2x^2 + a_3x^3) = a_0 + a_1(x+1) + a_2(x+1)^2 + a_3(x+1)^3$ for every member $a_0 + a_1x + a_2x^2 + a_3x^3 \in \rho_3[0,1]$; obtain representative matrix for T relative to basis (i) $(1, x, x^2, x^3)$ and (ii) $(1, 1+x, 1+x^2, 1+x^3)$ of $\rho_3[0,1]$

Solution : Here Dim $\rho_3[0,1] = 4$; So required matrix for linear operator T is of order 4×4; where $T : \rho_3[0,1] \rightarrow \rho_3[0,1]$.

Now (i) $(1, x, x^2, x^3)$ forms a basis for $\rho_3[0,1]$.

Now we have,

$T(1) = 1$, $T(x) = (x + 1)$, $T(x^2) = (x + 1)^2$ and $T(x)^3 = (x + 1)^3$. So we write with respect to basis above

$T(1) = 1 \quad = 1.1 + 0.x + 0.x^2 + 0.x^3$

$T(x) = 1 + x \quad = 1.1 + 1.x + 0.x^2 + 0.x^3$

$T(x^2) = (x + 1)^2 \quad = 1.1 + 2.x + 1.x^2 + 0.x^3$

$T(x^3) = (x + 1)^3 \quad = 1.1 + 3.x + 3.x^2 + 1.x^3$

Therefore representative matrix for T in this case shall be

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(ii) Here basis is $(1, 1+x, 1+x^2, 1+x^3)$ of $\rho_3[0,1]$

We have $T(1) = 1$, $T(1+x) = 1 + (1 + x)$, $T(1+x^2) = 1 + (1+x)^2$ and $T(1+x^3) = 1 + (1+x)^3$

Therefore relative to basis $(1, 1+x, 1+x^2, 1+x^3)$ we write

$T(1) \quad = 1 \ = 1.1 + 0.(1+x) + 0.(1+x^2) + 0.(1+x^3)$

$T(1+x) \ = 2+x \ = 1.1 + 1.(1+x) + 0.(1+x^2) + 0.(1+x^3)$

$T(1+x^2) = 1+1+2x+x^2 \ = -1.1 + 2.(1+x) + 1.(1+x^2) + 0.(1+x^3)$

$T(1+x^3) = 1+1+3x+3x^2+x^3 \ = -5.1 + 3.(1+x) + 3.(1+x^2) + 1.(1+x^3)$

Therefore representative matrix for T in this case shall be

$$\begin{pmatrix} 1 & 1 & -1 & -5 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Note : Basis taken and treated above should be termed as ordered basis. In ordered basis order of arrangement of vectors is basis in important. For example, in

Euclidean 3-space R^3 we know (e_1, e_2, e_3) is a basis in R^3, where $e_1 = (1,0,0)$, $e_2 = (0,1,0)$ and $e_3 = (0,0,1)$. Then each of (e_1, e_2, e_3), (e_2, e_1, e_3) and (e_1, e_3, e_2) is an ordered basis and they are different ordered basis for R^3.

§ 3.2(A) ISOMORPHIC LINEAR SPACES :

Definition 3.2.1. Two linear spaces X and Y over the same scalars are said to be isomorphic (or, linearly isomorphic) if there is a linear operator $T : X \to Y$ that is 1–1 (injective) and onto (surjective). The operator T is called an Isomorphism.

Theorem 3.2.1. Linear isomorphism between linear spaces over same scalars on the class Γ, of all such spaces is an equivalence relation.

Proof : If $X \in \Gamma$, the identity operator $I : X \to X$ is an isomorphism. So the binary relation of being isomorphic is reflexive; let $X, Y \in \Gamma$ such that X is isomorphic to Y with $\varphi : X \to Y$ as an isomorphism; Then $\varphi^{-1} : Y \to X$ is also an isomorphism. Thus Y is isomorphic to X. Hence relation of isomorphism is symmetric. Finally, if $f : X \to Y$ and $g : Y \to Z$ are isomorphism, then $(g . f) : X \to Z$ is also an isomorphism. Therefore, the relation of isomorphism is transitive. Thus it is an equivalence relation.

Theorem 3.2.2. Every real linear space X with $\dim(X) = n$ is isomorphic to the Euclidean n-space R^n.

Proof : Let $(u_1, u_2, ..., u_n)$ form a basis in X. So if $u \in X$ we write

$$u = \xi_1 u_1 + \xi_2 u_2 + ... + \xi_n u_n \text{ uniquely.}$$

Define an operator $T : X \to R^n$ by the rule :

$$T(u) = (\xi_1, \xi_2, ..., \xi_n) \in R^n \text{ where } u = \xi_1 u_1 + \xi_2 u_2 + ... + \xi_n u_n \in X$$

Then it is easily verified that T is a linear operator. Furhter, if $u = \sum_{i=1}^{n} \xi_i u_i$ and $v = \sum_{i=1}^{n} \eta_i u_i$ with $u \neq v$ are members of X, then we have

$$(\xi_1, \xi_2, ..., \xi_n) \neq (\eta_1, \eta_2, ..., \eta_n) \text{ or } T(u) \neq T(u);$$

thus T is 1–1. Finally, for $(\alpha_1, \alpha_2, ..., \alpha_n) \in R^n$

We have $\sum_{i=1}^{n} \alpha_i u_i \in X$ such that $T\left(\sum_{i=1}^{n} \alpha_i u_i\right) = (\alpha_1, \alpha_2, ..., \alpha_n)$.

So T is onto. Therefore X is isomorphic to R^n.

Notation : If two linear space X and Y are isomorphic we use the symbol $X \cong Y$.

Corollary : Any two real linear spaces of same finite dimension are isomorphic Because if X and Y are finite dimensional real linear spaces with Dim(X) = Dim(Y), we apply Theorem 3.2.2. to say $X \cong R^n$; and hence $X \cong Y$.

Theorem 3.2.3. Every linear operator over a finite dimensional *NLS* is bounded (hence continuous).

Proof : Let (X, || ||) and (Y, || ||) be two *NLS* over same scalars and Dim(X) $<\infty$, say, being equal to n, and let (e_1, e_2, ..., e_n) be a basis for X. Then each member $x \in X$ has a unique representative as $x = \xi_1 e_1 + \xi_2 e_2 + ... + \xi_n e_n$ where ξ_i's are scalars. Let us define a norm $\|x\|'$ by the formula :

$$\|x\|' = \sum_{i=1}^{n} |\xi_i|.$$

It is an easy task to check that $\|x\|'$ is indeed a norm in X. Since X is finite dimensional, we know that any two norms in X are equivalent.

Therefore there is a +ve M satisfying

$$\|x\|' \leq M\|x\| \text{ for all } x \in X$$

$$i.e. \quad \sum_{i=1}^{n} |\xi_i| \leq M\|x\| \qquad \text{........... (*)}$$

If $T : X \to Y$ is a linear operator and $x = \sum_{i=1}^{n} \xi_i e_i \in X$, we have

$$\|T(x)\| = \left\|T\left(\sum_{i=1}^{n} \xi_i e_i\right)\right\| = \left\|\sum_{i=1}^{n} \xi_i T(e_i)\right\|$$

$$\leq \sum_{i=1}^{n} |\xi_i| \|T(e_i)\|$$

$$\leq \max(\|T(e_i)\|, \|T(e_2)\|, ..., \|T(e_n)\|). M\|x\|$$

(from (*)) $= L\|x\|$, (say).

This being true for all $x \in X$, we conclude that T is bounded.

§ 3.3 SPACE OF ALL BOUNDED LINEAR OPERATORS Bd$\mathscr{L}$(X, Y)

Let $(X, \|\ \|)$ and $(Y, \|\ \|)$ be two *NLS* with same scalar field. Then zero operator $O : X \to Y$ where $O(x) = \underset{\sim}{0} \in Y$ *as* $x \in X$ is a bounded linear operator. Therefore $Bd\mathscr{L}(X,Y) \neq \phi$. It is a routine exercise to check that $Bd\mathscr{L}(X,Y)$ becomes a linear space with respect to addition and scalar multiplication as given by

$(T_1 + T_2)(x) = T_1(x) + T_2(x)$ for all $x \in X$; and $T_1, T_2 \in Bd\mathscr{L}(X,Y)$ and $(\lambda T_1)(x) = \lambda T_1(x)$ for all $x \in X$ and for all scalars λ and $T_1 \in Bd\mathscr{L}(X,Y)$

Theorem 3.3.1. $Bd\mathscr{L}$ *(X,Y)* is a Normed Linear space, and it is a Banach space when Y is so.

Proof : Let us take the norm in linear space $Bd\mathscr{L}(X,Y)$ as operator norm $\|T\|$ as $T \in Bd\mathscr{L}(X,Y)$. We verify that all norm axioms are satisfied here.

For (N.1) it is obvious that $\|T\| \geq 0$ always for any member $T \in Bd\mathscr{L}(X,Y)$; zero operator O has the norm $\|O\| = 0$.

Suppose $\|T\| = 0$ *i.e.* $\sup_{\|x\| \leq 1} \|T(x)\| = 0$. So if $\|x\| \leq 1$, we have

$$\|Tx\| \leq \sup_{\|x\| \leq 1} \|T(x)\| = 0 \text{ gives } \|T(x)\| = 0 \quad \ldots\ldots (1)$$

If $\|x\| > 1$, then put $y = \dfrac{x}{\|x\|}$; Thus $\|y\| = \left\| \dfrac{x}{\|x\|} \right\| = 1$; so as got above

$\|T(y)\| = 0$ or $0 = \|T(y)\| = \left\| T\left(\dfrac{x}{\|x\|}\right) \right\| = \dfrac{1}{\|x\|} \|T(x)\|$ giving

$$\|T(x)\| = 0 \quad \ldots\ldots (2)$$

So (1) and (2) say that $T(x) = \underset{\sim}{0}$ for all $x \in X$ *i.e.* T equals to the zero operator. For (N.2) take λ to be any scalar.

Then $\|\lambda T\| = \sup_{\|x\| \leq 1} \|(\lambda T)(x)\|$

$$= \sup_{\|x\| \leq 1} \|\lambda T(x)\| = \sup_{\|x\| \leq 1} \{|\lambda| \|T(x)\|\}$$

$$= |\lambda| \sup_{\|x\| \leq 1} \|T(x)\| = |\lambda| \|T\|.$$

So (N.2.) is satisfied.

For triangle inequality, if T_1, T_2 are members of $Bd\mathscr{L}(X,Y)$ we have for

$x \in X$, $\|T_1 + T_2)(x)\| = \|T_1(x) + T_2(x)\| \le \|T_1(x)\| + \|T_2(x)\|$

$\le \|T_1\|(x)\| + \|T_2\|\|x\| = (\|T_1\| + \|T_2\|)\|x\|$; this is true for all $x \in X$,

Therefore $\|T_1 + T_2\| \le \|T_1\| + \|T_2\|$, and that is the triangle inequality.

Therefore $Bd\mathscr{L}(X,Y)$ is a Normed Linear space (*NLS*) with respect to operator norm.

Now suppose that Y is a Banach space. We show that $Bd\mathscr{L}(X,Y)$ is so. Take $\{T_n\}$ as a Cauchy sequence in $Bd\mathscr{L}(X,Y)$ *i.e.* $\|T_n - T_m\| \to 0$, as $n, m \to \infty$

If $x \in X$, we have $\|T_n(x) - T_m(x)\| = \|(T_n - T_m)(x)\|$

$\le \|T_n - T_m\|\|x\| \to 0$ as $n, m \to \infty$. That means, $\{T_n(x)\}$ is a Cauchy sequence in (Y, $\|\;\|$) which is complete.

Let $\lim_{n\to\infty} T_n(x) = y \in Y$

Let us define $T : X \to Y$ by the rule :

$$T(x) = \lim_{n\to\infty} T_n(x) \text{ as } x \in X.$$

Now it is easy to see that T is a linear operator.

Further, $\left|\|T_n\| - \|T_m\|\right| \le \|T_n - T_n\| \to 0$ as $n, m \to \infty$.

That means $\{\|T_n\|\}$ is a sequence of non-negative reals and this is Cauchy sequence and therefore is bounded. So we find a +ve K satisfying

$$\|T_n\| \le K \text{ for all } n.$$

So, $\|T(x)\| = \|\lim_{n\to\infty} T_n(x)\| = \lim_{n\to\infty} \|T_n(x)\|$

$$\le \lim_{n\to\infty} \|T_n\|\|x\| \le K\|x\| \text{ by above inequality.}$$

This being true for all $x \in X$, we find $T : X \to Y$ as a bounded linear operator *i.e.* $T \in Bd\mathscr{L}(X,Y)$.

Finally, from Cauchyness of $\{T_n\}$, given a +ve ε, we have

$$\| T_{n+p} - T_n \| < \varepsilon \text{ for } n \geq n_0 \text{ and } p = 1,2,......$$

Take $\| x \| \leq 1$ in X, So $\| T_{n+p}(x) - T_n(x) \| = \| (T_{n+p} - T_n)(x) \|$

$$\leq \| T_{n+p} - T_n \| \| x \| \leq \| T_{n+p} - T_n \| . < \varepsilon \text{ for } n \geq n_0$$

Let us pass on limit as $p \to \infty$, then we have

$$\| T(x) - T_n(x) \| \leq \varepsilon \text{ whenever } n \geq n_0$$

This is the case whenever $\| x \| \leq 1$; taking sup we have

$$\sup_{\| x \| \leq 1} \| T(x) - T_n(x) \| \leq \varepsilon \text{ whenever } n \geq n_0$$

Now $\| T - T_n \| = \sup\limits_{\| x \| \leq 1} \| (T - T_n)(x) \|$

$$= \sup_{\| x \| \leq 1} \| (T(x) - T_n(x) \|$$

$$\leq \varepsilon \text{ whenever } n \geq n_0$$

So we obtain $\lim\limits_{n \to \infty} T_n = T \in Bd\mathscr{L}(X, Y)$ in operator norm.

The proof is now complete.

Example 3.3.1. Show $Bd\mathscr{L}(R^n, R^n)$ is finite dimensional with dimension n^2.

Solution : By matrix representation theorem we know that every member $T \in Bd\mathscr{L}(R^n, R^n)$ has a representative matrix of order $n \times n$ (*i.e.* a square matrix of size n). With respect to a fixed basis in R^n, we also see that $Bd\mathscr{L}(R^n, R^n)$ and the linear space $m_{n \times n}$ is finite dimensional with $\text{Dim}(m_{n \times n}) = n^2$.

Therefore $\text{Dim}(Bd\mathscr{L}(R^n, R^n)) = n^2$

Example 3.3.2. A *NLS* (X, $\| \ \|$) is a Banach space if and only if $\{x \in X : \| x \| = 1\}$ is complete.

Solution : Suppose (X, $\| \ \|$) is a Banach space; then the given set $\{x \in X : \| x \| = 1\}$ is a closed subset of X, and hence is complete.

Conversely, suppose $S=\{x\in X:\|x\|=1\}$ is complete. Now let $\{x_n\}$ be a Cauchy sequence in X, so $\|x_n-x_m\|\to 0$ as $n,m\to\infty$

Therefore $\big|\|x_n\|-\|x_m\|\big|\le\|x_n-x_m\|\to 0$ as $n,m\to\infty$. Thus scalar sequence $\{\|x_n\|\}$ is Cauchy, and by Cauchy General Principle of convergence $\{\|x_n\|\}$ is convergent; put $\lim_{n\to\infty}\|x_n\|=\alpha$. If $\alpha = 0$ we see $\{x_n\}$ to be convergent in X and we have finished. Or else $\alpha > 0$. Without loss of generality we assume that $\alpha = 1$. Let us put $y_n=\dfrac{x_n}{\|x_n\|}$ making $\|y_n\|=1$ *i.e.* $y_n\in S$. If possible, let $\{y_n\}$ be not Cauchy. Then there is a +ve ε_0 (say) and there are indices $n_k(\ge k), m_k(\ge k)$ such that

$$\|y_{n_k}-y_{m_k}\|\ge\varepsilon_0,\ \ k=1,2,\ldots\ldots$$

$$\text{or, } \varepsilon_0\le\left\|\frac{x_{n_k}}{\|x_{n_k}\|}-\frac{x_{m_k}}{\|x_{m_k}\|}\right\|\le\left\|\frac{x_{n_k}}{\|x_{n_k}\|}-x_{n_k}\right\|+\|x_{n_k}-x_{m_k}\|+\left\|x_{m_k}-\frac{x_{m_k}}{\|x_{m_k}\|}\right\|$$

$$=\|x_{n_k}\|\left|1-\frac{1}{\|x_k\|}\right|+\|x_{m_k}\|\left|1-\frac{1}{\|x_{m_k}\|}\right|\to 0 \quad \text{as} \quad k\to\infty;$$ arriving at contradiction that ε_0 is +ve. Therefore we conclude that $\{y_n\}$ is Cauchy in S by completeness of which let $\lim_{n\to\infty}y_n=y_0\in S$. That is $\lim_{n\to\infty}x_n=\lim_{n\to\infty}\|x_n\|\,y_0=y_0$.

Hence $\{x_n\}$ is convergent in X and X is shown as a Banach space.

EXERCISE A

Short answer type questions

1. Let X be the linear space spanned by f and g where $f(x) = \sin x$ and $g(x) = \cos x$. For any real θ, let $f_1(x) = \sin(x+\theta)$ and $g_1(x) = \cos(x+\theta)$. Show that f_1 and g_1 are members of X, and they are linearly independent.
2. Let A and B be two subsets of a *NLS* X and let $A+B=\{a+b: a\in A \text{ and } b\in B\}$. Show that if A or B is open then $A + B$ is open.
3. Let $m_{2\times 2}$ be the linear space of all real 2×2 matrics and $E=\begin{pmatrix}1 & -1\\ 0 & 0\end{pmatrix}$.

If $T : m_{2\times2} \to m_{2\times2}$ is taken as $T(A) = EA$ for $A \in m_{2\times2}$, show that T is a linear operator.

4. If C is a convex subset of a *NLS* X and $x_0 \in X$, and α is a non-zero scalar, show that $x_0 + C$ and αC are convex sets.
5. Show that $T : C[a,b] \to R$ (real space with usual norm) defined by the rule :

$$T(f) = \int_a^b tf(t)dt \;; \qquad f \in C[a,b].$$

Show that T is a bounded linear operator.

EXERCISE B

1. Let A and B be two subsets of a *NLS* X, and let $A + B = \{a + b : a \in A \text{ and } b \in B\}$. If A and B are compact, show that $A + B$ is compact.
2. Let M be a closed linear sub-space of a *NLS* $(X, \|\ \|)$, and X/M be the quotient space, and $T : X \to X/M$ where $T(x) = x + M$ for $x \in X$.

   Show that T is a bounded linear operator with $\|T\| \le 1$.
3. Show that the space of all real polynomials of degree $\le n$ is the closed interval $[a,b]$ is isomorphic to the Euclidean $(n+1)$-space R^{n+1}.
4. Let $(X, \|\ \|)$ and $(Y, \|\ \|)$ be *NLS* over same scalars and F, $T : X \to Y$ be bounded linear operators such that F and T agree over a dense set in X, show that $F \equiv T$.
5. If X is a finite Dimensional *NLS*, and Y is a proper sub-space of X, then show that there is a member $x \in X$ with $\|x\| = 1$. satisfying dist$(x, Y) = 1$.