# UNIT 4

(***Contents*** : Bounded Linear functionals, sub-linear functionals, Hahn-Banach Theorem; Its applications, Conjugate spaces of a *NLS*, Canonical mapping, Embedding of a *NLS* into its second conjugate space under a linear isometry, reflexive Banach space; Open mapping theorem, Closed Graph Theorem.).

## § 4.1 LINEAR FUNCTIONALS :

Let ($X$, || ||) be a *NLS* over reals/complex numbers.

**Definition 4.1.1.** A Scalar-valued Linear operator $f$ over $X$ is called a Linear functional.

For example if $X$ = Banach space $C[0,1]$ with sup norm, then $f : X \to$ Reals (with usual norm) is a linear functional when $f(x) = \int_0^1 x(t)dt; \quad x \in C[0,1]$.

**Explanation** : Linear functionals are special kind of Linear operators, and thus enjoy all the properties of Linear operators like sending dependent set of the domain into a similar such elements in range.

Let us consider the collection of all continuous (bounded) linear functionals over $X$ *i.e.* we have the space $Bd\mathscr{L}(X,R)$ whenever $X$ is a real *NLS*. We have seen that the space $Bd\mathscr{L}(X,R)$ is always a *NLS* with operator norm $||f||$; $f$ being a member of $Bd\mathscr{L}(X,R)$. We have also seen that the *NLS* $Bd\mathscr{L}(X,R)$ is a Banach space because $R$ is so.

**Definition 4.1.2.** The space $Bd\mathscr{L}(X,R)$ denoted by $X^*$ is called first conjugate space (Dual space) of $X$.

Thus first conjugate space or simply conjugate space $X^*$ of any *NLS* ($X$, || ||) is always a Banach space irrespective of $X$ being complete or not.

By a similar construction one can produce $Bd\mathscr{L}(X^*,R)$ = the space of all bounded linear functionals over $X^*$; this Banach space $X^{**} = (X^*)^*$ is called second conjugate (Dual) space of $X$; and so on.

Most of theory of conjugate spaces rests on one single theorem, known as famous Hahn-Banach Theorem that asserts that any continuous linear functional on a linear subspace of $X$ can be extended to a continuous linear functional over $X$ by keeping the norm-value of the functional unchanged. The proof of Hahn-Banach Theorem is lengthy but necessarily indispensable item in Functional Analysis.

Before we take up Hahn-Banach Theorem in setting of a *NLS* we proceed as under :

**Definition 4.1.2.** Let $X$ be a real linear space. Then $p : X \to$ Reals satisfying (i) $p(x+y) \le p(x) + p(y)$ for all $x, y \in X$ and (ii) $p(\alpha x) = \alpha p(x)$ for all $\alpha \ge 0$, $x \in X$ is called a sub-linear functional.

**Note** : Condition (i) above is known as condition of sub-additivity and condition (ii) above is called positive homogeneity.

It is not difficult to see that norm function in a *NLS* $X$ is a sub-linear functional over $X$.

**Theorem 4.1.1. (Hahn-Banach Theorem in a linear space)**

Let $M$ be a subspace of a real linear space $X$, and $p$ is a sub-linear functional over $X$ and $f$ is a linear functional on $M$ such that $f(x) \le p(x)$ for all $x \in M$.

Then there is a linear functional $F$ over $X$ which is an extension of $f$ (over $M$) such that

$$F(x) \le p(x) \text{ for all } x \in X.$$

The proof of this Theorem rests upon following Lemma.

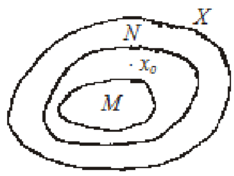
**Lemma 4.1.1.** Suppose $M$ is a subspace ($\ne X$) of a real linear space $X$ and $x_0 \in (X \backslash M)$. Let $N$ be the subspace spanned by $M$ and $\{x_0\}$ *i.e.* $N = [M \cup \{x_0\}]$; suppose $f : M \to R$ is a Linear functional such that

$f(x) \le p(x)$ for all $x \in M$, where $p : X \to R$ is a sub-linear functional (over $X$).

Then $f$ can be extended to a linear functional $F$ defined on $N$ such that

$$F(x) \le p(x) \quad \text{for } x \in N.$$

**Proof :** Since $f(x) \le p(x)$ over $M$, we have for $y_1, y_2 \in M$.




$$f(y_1 - y_2) = f(y_1) - f(y_2) \le p(y_1 - y_2) = p(y_1 + x_0 - y_2 - x_0)$$
$$\le p(y_1 + x_0) + p(-y_2 - x_0)$$

or, $-p(-y_2 - x_0) - f(y_2) \le p(y_1 + x_0) - f(y_1)$ ........................................ (1)

(separation of terms involving $y_1$ and $y_2$)

Now fix $y_1$ and allow $y_2$ to change over $M$. From (1) we see that the set of reals $\{-p(-y_2 - x_0) - f(y_2)\}$ possesses sup.

Put $a = \sup\limits_{y_2 \in M} \{-p(-y_2 - x_0) - f(y_2)\}$; and in a similar argument, put

$b = \operatorname{Inf}_{y_1 \in M}\{p(y_2 + x_0) - f(y_1)\}$. The relation (1) says, $a \le b$.

Take a real $c_0$ between $a$ and $b$ *i.e.* $a \le c_0 \le b$

Therefore as $y \in M$ we have

$$-p(-y - x_0) - f(y) \le c_0 \le p(y + x_0) - f(y) \qquad \text{................ (2)}$$

Since $x_0 \notin M$, we write $x_0 \in N$ as $x = y + \alpha x_0$, and this representation is unique.

Consider $F : N \to R$ defined by the rule :

$F(y + \alpha x_0) = f(y) + \alpha c_0$, as $(y + \alpha x_0) \in N$ ($y \in M$ & $\alpha$ a scalar). It is easy to check that $F$ is a linear functional over $N$ such that $F(y) = f(y)$ as $y \in M \subset N$.

In other words $F$ is an extension of $f$ from $M$ to $N$. We verify further that

$F(x) \le p(x)$ for all $x \in N$. To achieve this we are to consider following two cases : When $x \in N$, we have $x = y + \alpha x_0$, where $\alpha$ is a scalar.

**Case I** . When $\alpha > 0$; we consider R.H.S. of inequality (2) with $y$ replaced by $\frac{y}{\alpha}$; this gives $c_0 \le p\left(\frac{y}{\alpha} + x_0\right) - f\left(\frac{y}{\alpha}\right)$.

Multiplying throughout by $\alpha$ and using the fact that $p$ is sub-linear we have

$$f(y) + \alpha c_0 \le p(y + \alpha x_0)$$

$$\text{or,} \quad F(x) \le p(x)$$

**Case II**. When $\alpha < 0$, we use L.H.S. of inequality (2) with $y$ replaced by $\frac{y}{\alpha}$. This gives rise to

$$-p\left(-\frac{y}{\alpha} - x_0\right) - f\left(\frac{y}{\alpha}\right) \le c_0$$

$$\text{or,} \qquad -p\left(\frac{y}{\alpha} - x_0\right) \le c_0 + f\left(\frac{y}{\alpha}\right).$$

Multiplying throughout by $\alpha$ and reversing the sign we have,

$$(-\alpha)p\left(-\frac{y}{\alpha} - x_0\right) \ge \alpha c_0 + f(y)$$

Since – α > 0, we have $p(y+\alpha x_0) \geq \alpha c_0 + f(y)$

or, $p(x) \geq F(x)$

or, $F(x) \leq p(x)$

When α = 0, we readily see $F(y) = f(y)$. The proof of Lemma is now complete.

**Proof of Theorem 4.1.1.** To prove the theorem we invite partial ordering in a set and use Zorn's Lemma which says that in a partially ordered set if every chain has an upper bound, then there is a maximal member in the set.

Here let Γ denote the collection of all linear functionals $\{\hat{f}\}$ such that each $\hat{f}$ is an extension of *f* such that $\hat{f}(x) \leq p(x)$ over domain of $\hat{f} \equiv D_{\hat{f}}$.

Lemma 4.1.1 tells us that Γ is non-empty. Let us partially order Γ as for $\hat{f}_1, \hat{f}_2 \in \Gamma$ we say, $\hat{f}_1 < \hat{f}_2$

if $\hat{f}_2$ is an extension of $\hat{f}_1$ with $D_{\hat{f}_2} \supset D_{\hat{f}_1}$, and $\hat{f}_2 = \hat{f}_1$ over $D_{\hat{f}_1}$.

We may verify that $\alpha$ is a partial order relation in Γ where we show that every chain (totally ordered subset) in Γ has an upper bound in Γ. To that goal, let $\tau = \{\hat{f}_\alpha\}$ be a totally ordered subset of Γ. We find some member $\hat{f} \in \Gamma$ to act as an upper bound for $\tau$.

Construct $\hat{f}$ whose domain = $\underset{\alpha}{\cup} D_{\hat{f}_\alpha}$. If $x \in \underset{\alpha}{\cup} D_{\hat{f}_\alpha}$ there is a member $\alpha$ such that $x \in D_{\hat{f}_\alpha}$ and let $\hat{f}(x) = f_{\hat{\alpha}}(x)$

By routine work we verify that $\underset{\alpha}{\cup} D_{\hat{f}_\alpha}$ is a sub-space of *X*; taking $x, y \in \underset{\alpha}{\cup} D_{\hat{f}_\alpha}$ we find two indices $\alpha_1$ and $\alpha_2$ such that $x \in D_{\hat{f}_{\alpha_1}}$ and $y \in D_{\hat{f}_{\alpha_2}}$.

Since $\tau$ is totally ordered either $D_{\hat{f}_{\alpha_1}} \subset D_{\hat{f}_{\alpha_2}}$ or $D_{\hat{f}_{\alpha_2}} \subset D_{\hat{f}_{\alpha_1}}$, and in either of the cases we have

$(x+y) \in \underset{\alpha}{\cup} D_{\hat{f}_\alpha}$ and similarly $\alpha x \in \underset{\alpha}{\cup} D_{\hat{f}_\alpha}$ and $\underset{\alpha}{\cup} D_{\hat{f}_\alpha}$ is a sub-space of *X*.

Finally we show $\hat{f}$ is well-defined.

Suppose $x \in D_{\hat{f}_\alpha}$ and $x \in D_{\hat{f}_\beta}$; by definition

$\hat{f}(x) = f_{\hat{\alpha}}(x)$ and $\hat{f}(x) = f_{\hat{\beta}}(x)$

By total ordering of $\tau$ either $\hat{f}_\alpha$ is an extension of $\hat{f}_\beta$ or vice-versa.

So $\hat{f}_\alpha(x) = \hat{f}_\beta(x)$. Thus we have

$\hat{f}(x) \le p(x)$ for $x \in D_{\hat{f}}$ and for any member $\hat{f}_\alpha$ of $\tau$, we have $\hat{f}_\alpha \alpha \hat{f}$. So $\hat{f} \in \Gamma$ is an upper bound of $\tau$. So we apply Zorn's Lemma to obtain a maximal member (say) $F$ in $\Gamma$. And $F$ is the desired extension of $f$ as a linear functional with $F(x) \le p(x)$ for all $x \in X$ ; that domain of $F$ equals to $X$ follows maximality of $F$; Otherwise by argument as above one can have an extension of $F$ to some other functional—a contradiction of maximality of $F$. The proof of theorem is now complete.

**Remark :** Theorem 4.1.1 is also true for complex spaces, for which one has to furnish proof.

**Theorem 4.1.2. (Hahn-Banach Theorem in a *NLS*).**

Suppose $f$ is a bounded linear functional on a sub-space $M$ of *NLS* $X$. There is a bounded linear functional $F$ which is an extension of $f$ from $M$ to $X$ having the same norm as that of $f$.

**Proof :** If $x \in M$ we have $|f(x)| \le \|f\| \|x\|$.

Define $p: X \to R$ by the rule :

$$p(x) = \|f\| \|x\| \text{ for } x \in X.$$

Then we verify that $p$ is a sub-linear functional over $X$.

Such that $f(x) \le p(x)$ for $x \in M$.

Now apply Theorem 4.1.1 (Hahn-Banach Theorem in real space) to get a linear functional $F$ which is an extension of $f$ from $M$ to $X$ such that

$$|F(x)| \le p(x) \quad \text{for all } x \in X.$$

$$\text{i.e. } |F(x)| \le \|f\| \|x\| \quad \text{for all } x \in X.$$

This is true for all $x \in X$ ; So we conclude that $F$ is a bounded linear functional over $X$ with $\|F\| \le \|f\|$ ............... (1)

Further, over $M$ we have $f(x) = F(x)$

So $|f(x)| = |F(x)| \le \|F\| \|x\|$ for all $x \in M$. This gives

$$\|f\| \le \|F\| \qquad \text{............... (2)}$$

Now (1) and (2) together say $\|f\| = \|F\|$

## § 4.2 SOME CONSEQUENCES OF HAHN-BANACH THEOREM :

**Application I.** Given a real *NLS* $(X, \| \|)$ and a non-zero member $x_0 \in X$. There is a bounded linear functional $F$ over $X$ such that $F(x_0) = \| x_0 \|$ with $\| F \| = 1$.

**Proof :** Consider the sub-space $M$ of $X$ spanned by $x_0$;

Then $M = [x_0] = \{\alpha x_0 : \alpha$ any real scalar$\}$

Define $f : M \to$ Reals by the rule :

$$f(\alpha x_0) = \alpha \| x_0 \|; \text{ as } (\alpha x_0) \in M.$$

Then $f$ is a linear functional over $M$ and $| f(x) | = | \alpha | \| x_0 \| = \| \alpha x_0 \|$ for all $x = \alpha x_0 \in M$ and hence we have $\| f \| \le 1$. *i.e.* $f$ is a bounded linear functional.

Further if $u = \alpha x_0$ is a member of $M$ with $\| u \| = 1$ we have

$$| f(u) | = | \alpha | \| x_0 \| = \| \alpha x_0 \| = \| u \| = 1$$

$$\therefore \ \| f \| \ge | f(u) | = 1 \text{ giving } \| f \| = 1.$$

Now an application of Hahn-Banach Theorem gives a bounded linear functional $F$ over $X$ satisfying

$$F(x) = f(x) \quad x \in M$$

and $$\| F \| = \| f \| = 1$$

This gives $F(x_0) = f(x_0) = \| x_0 \|$ and $\| F \| = 1$.

**Corollary** : For a non-null *NLS* $(X, \| \|)$, its conjugate space $X^*$ is non-null. (Hints : because $F$ appearing in corollary is non-zero member of $X^*$).

**Application II**. For every $x \in X$, $\| x \| = \sup\limits_{f (\neq 0) \in X^*} \dfrac{| f(x) |}{\| f \|}$.

**Proof :** From Application I we find a non-zero bounded linear functional $f_0 \in X^*$ such that $f_0(x) = \| x \|$ and $\| f_0 \| = 1$.

Therefore, $$\sup_{f \neq 0 \in X^*} \frac{| f(x) |}{\| f \|} \ge \frac{| f_0(x) |}{\| f_0 \|} = \| x \|$$

*i.e.* $$\sup_{f (\neq 0) \in X^*} \frac{| f(x) |}{\| f \|} \ge \| x \| \qquad \text{............... (1)}$$

On the other hand, if $f$ is any non-zero member of $X^*$, we have

$$|f(x)| \le \|f\|\|x\|$$

or $\dfrac{|f(x)|}{\|f\|} \le \|x\|$, r.h.s. being independent of $f$

we have, $$\sup_{f(\neq 0)\in X^*} \frac{|f(x)|}{\|f\|} \le \|x\| \qquad \text{.............. (2)}$$

From (1) and (2) one has $\|x\| = \sup_{f(\neq 0)\in X^*} \dfrac{|f(x)|}{\|f\|}$.

**Corollary :** If $f(x) = 0$ for every non-zero bounded linear functional $f \in X^*$, then $x = \underset{\sim}{0}$ in $X$.

**Application III.** Let $M$ be a closed subspace of $X$ and $M \neq X_0$, if $u \in (X \setminus M)$ and $d = dist(u, M) = \underset{m\in M}{Inf} \|u - m\|$;

Then $d > 0$, and there is a bounded linear functional $f \in X^*$ such that

(i) $f(x) = 0$ for $x \in M$

(ii) $f(u) = 1$

and (iii) $\|f\| = \dfrac{1}{d}$.

**Proof :** Here $M$ is a closed sub-space ($\neq X$); so $d > 0$.

Take $N$ = Linear subspace spanned by $M$ and $u$

*i.e.* $N = [M \cup \{u\}]$; So every member of $N$ is of the form $m + tu$ where $t$ is a real scalar, and $m \in M$.

Define $g : N \to R$ by the rule :

$$g(m + tu) = t \text{ as } (m + tu) \in N.$$

It is easy to check that $g$ is a linear functional over $N$ such that $g$ vanishes over $M$ *i.e.* $g(m) = 0$ for $m \in M$, and $g(u) = 1$ (taking $t = 1$).

Now $|g(m + tu)| = |t| = \dfrac{|t|\|m + tu\|}{\|m + tu\|} = \dfrac{\|m + tu\|}{\|\frac{m}{t} + u\|}$

$$= \frac{\| m + tu \|}{\| u - (\frac{m}{t}) \|} \le \frac{\| m + tu \|}{d} = \frac{1}{d} \| m + tu \| .$$

because $d = \inf_{v \in m} \| u - v \| \le \| u - (-\frac{m}{t}) \|$.

This is true for all member $(m + tu) \in N$; and hence $g$ is a bounded linear functional over $N$ with $\| g \| \le \frac{1}{d}$.

So, $\| g \| \le \frac{1}{d}$ ................ (1)

Again from $d = \inf_{v \in m} \| u - m \|$; we find a sequence $\{m_n\}$ in $M$

such that $\| u - m_n \| \to d$ as $n \to \infty$

*i.e.* $\lim_{n \to 0} \| u - m_n \| = d$ ................ (2)

Now $| g(m_n - u) | \le \| g \| \| m_n - u \|$

or, $| g(m_n) - g(u) | \le \| g \| \| m_n - u \|$

or, $| 0 - 1 | \le \| g \| \| m_n - u \|$; ($g$ vanishing over $M$ and $g(u)$ = 1).

or, $1 \le \| g \| \| m_n - u \|$

Now passing on limit as $n \to \infty$ we produce

$$1 \le \| g \| . d$$

giving, $\| g \| \ge \frac{1}{d}$ ................ (3)

Combining (1) and (3) we have $\| g \| = \frac{1}{d}$.

Finally, Hahn-Banach Theorem says that $g$ has an extension $f$ from $N$ to the whole space $X$ as a bounded linear functional with $\| f \| = \| g \|$; As $f$ and $g$ agree over $M \subset N$, we have the result as wanted.

**Application IV.** Let $M$ be a sub-space of *NLS* ($X$, $\| \, \|$) and $M \ne X$; if $u \in (X \setminus M)$ such that *dist*($u$, $M$) > 0, say = $d$.

Then there is a bounded linear functional $F \in X^*$ satisfying

(i) $F(x) = 0$ over $M$ (for $x \in M$)

(ii) $F(u) = d$

and (iii) $\| F \| = 1$.

**Proof :** Let $N$ = Linear sub-space spanned by $M$ plus $u$, *i.e.* $N = [M \cup \{u\}]$

Now define $f : N \to$ Reals by rule :

$f(m + tu) = td$ ($d$ as above), where $m + tu$ is a representative member of $N$($m \in M$, $t$ a scalar).

Clearly $f$ is a linear functional over $N$, such that for $t = 0$, $f$ vanishes over $M$ and $f(u) = d$ ($t = 1$).

Also for $t \neq 0$, $\| m + tu \| = \| -t\left(-\frac{m}{t} - u\right) \|$ (here $\frac{-m}{t} \in M$ )

$$= |t| \; \| -\frac{m}{t} - u \| \geq |t| \, d.$$

So, $| f(m + tu) = |t| \, d \leq \| m + tu \|$; this inequality stands even for $t = 0$.

That means, $f$ is a bounded linear functional over $N$ with $\| f \| \leq 1$.

For $\in > 0$, we find by Infimum property, a member $m \in M$ such that $\| m - u \| < d + \in$.

Put $v = \frac{m - u}{\| m - u \|}$, making $\| v \| = 1$ and $v \in N$ (because, $v$ is the form $m' + t'u$).

So, $| f(v) | = \frac{d}{\| m - u \|} > \frac{d}{d + \in} = \frac{d}{d + \in} \| v \|$ $(\because \| v \| = 1)$

That means, $\| f \| \geq \frac{d}{d + \in}$. Now this is true for every +ve $\in$, and taking $\in \to 0_+$, we find $\| f \| \geq 1$.

*i.e.* $\| f \| \geq 1$ ................ (2)

Combining (1) and (2) we find $\| f \| = 1$. Now we apply Hahn-Banach Theorem to find an extension $F$ of $f$ from $N$ to the whole space $X$ as a bounded linear functional over $X$ with $\| F \| = \| f \|$; since $F$ agrees with $f$ over $M$, we have the result as desired.

## § 4.3 CONJUGATE SPACES *X**, *X***, ... OF A *NLS* (*X*, || ||) :

Let (*X*, || ||) be a *NLS*; then *X**, *X*** = (*X**)*, .. are first, second, ...conjugate space of *X*.

**Theorem 4.3.1.** If *X** is separable, then so is *X*.

**Proof :** Suppose $D$ is a countable dense subset of *X**. Let $D_1$ be the subset of $D$ which is dense in the surface $\{f \in X^* : \|f\| = 1\}$ of the closed unit ball of *X**; let us write $D_1 = \{f_1, f_2, ..., f_n, ...\}$ with $\|f_n\| = 1$ for all *n*. From $\|f_n\| = 1$, we find a member say $x_n$ with $\|x_n\| = 1$ such that

$$|f_n(x_n)| > \frac{1}{2}.$$

Consider the linear sub-space $L$ of $X$ spanned by $\{x_1, x_2, ...., x_n\}$

*i.e.* $L = [x_1, x_2, ...., x_n .....]$ and Put $M = \overline{L}$ (closure of $L$). The $M$ is also a linear sub-space of $X$.

Suppose, $M \neq X$ ................ (1)

Take $x_0 \in (X \setminus M)$, then $d = dist(x_0, M) > 0$ because $M$ is closed.

By application of Hahn-Banach Theorem we obtain a bounded linear functional $F \in X^*$ with $\|F\| = 1$ such that $F$ vanishes ($F$ = 0) over $M$ and $F(x_0) \neq 0$.

Clearly $F$ is a member of the set $\{f \in X^* : \|f\| = 1\}$ and $F(x_n) = 0$ for all *n*.

Now $f_n(x_n) = f_n(x_n) - F(x_n) + F(x_n)$ gives

$$|f_n(x_n)| \leq |f_n(x_n) - F(x_n)| + |F(x_n)|$$

$$= |(f_n - F)(x_n)|$$

Thus $\frac{1}{2} < |f_n(x_n)| \leq \|f_n - F\| \|x_n\|$

or, $\frac{1}{2} < \|f_n - F\|$ for all *n*; This contradicts that $\{f_1, f_2, ..., f_n, ..\}$ is dense in the set $\{f \in X^* : \|f\| = 1\}$.

So, $M = X$.

That is $\overline{L} = X$;

Now $L$ contains that subset formed by finite linear combinations of $x_1, x_2,.., x_n,..$ with rational coefficients; and that subset becomes countable dense in $X$. The proof is now complete.

**Remark :** Converse of Theorem 4.3.1 is not true. The *NLS* $l_1$ consisting of all those real sequences $\underset{\sim}{x} = (x_1, x_2 ... x_n, ....)$ such that $\sum_{i=1}^{\infty} |x_i| < \infty$ with norm $\|\underset{\sim}{x}\| = \sum_{i=1}^{\infty} |x_i|$ is separable but its conjugate space $l_\infty$ consisting of all bounded sequences of reals is not separable.

**Example 4.3.1.** Let $(X, \|\ \|)$ be a *NLS* over reals, and let $x_1, x_2 \in X$ with $x_1 \neq x_2$. Show that there is a bounded linear functional $f$ over $X$ such that $f(x_1) \neq f(x_2)$.

**Solution :** Here $x_1, x_2 \in X$ with $x_1 \neq x_2$ *i.e.* $x_1 - x_2 \neq \underset{\sim}{O}$ in $X$. So an application of Hahn-Banach Theorem there is a bounded linear functional $f \in X^* (f \neq 0)$ such that

$$f(x_1 - x_2) \neq 0$$

or, $$f(x_1) - f(x_2) \neq 0$$

or, $$f(x_1) \neq f(x_2).$$

Given a *NLS* $(X, \|\ \|)$ we show that there is a natural embedding of $X$ in its second conjugate space $X^{**}$ through a mapping, called the Canonical mapping that we presently define using $X^*$.

**Theorem 4.3.2.** Given $x \in X$, let $\hat{x}(x^*) = x^*(x)$ for all $x^* \in X^*$. Then $\hat{x}$ is a bounded linear functional over $X^*$, and the mapping $x \to \hat{x}$ is a Linear Isometry of $X$ into $X^{**}$.

**Proof :** Let $x \in X$, $x_1^*, x_2^* \in X^*$; then we have

$$\hat{x}(x_1^* + x_2^*) = (x_1^* + x_2^*)(x) = x_1^*(x) + x_2^*(x) = \hat{x}(x_1^*) + \hat{x}(x_2^*).$$

Also if $\lambda$ is any scalar we have $\hat{x}(\lambda x_1^*) = (\lambda x_1^*)(x) = \lambda x_1^*(x) = \lambda \hat{x}(x_1^*)$.

Therefore $\hat{x}$ is a linear functional over $X^*$.

Now we show that $\|x\| = \sup_{\|x^*\| \leq 1} \{|x^*(x)|\}$.

By Hahn-Banach Theorem we find a member $x^* \in X^*$ with $\|x^*\| = 1$ and $\|x^*(x)\| = \|x\|$.

Therefore $\|x\| \leq \sup_{\|x^*\|\leq 1} \{|x^*(x)|\}$ ................ (1)

Again $\|x^*(x)\| \leq \|x^*\|\|x\| \leq \|x\|$ when $\|x^*\| \leq 1$

Therefore $\|x\| \geq |x^*(x)|$ when $\|x^*\| \leq 1$

Thus $\|x\| \geq \sup_{\|x^*\|\leq 1} |x^*(x)|$. ................ (2)

From (1) and (2) we have

$$\|x\| = \sup\{|x^*(x)| : x^* \in X^* \text{ with } \|x^*\| \leq 1\}.$$

which is $= \sup\{|\hat{x}(x^*)| : x^* \in X^* \text{ with } \|x^*\| \leq 1\}$

$$= \|\hat{x}\|.$$

It shows that $\hat{x}$ is a bounded linear functional over $X^*$ with $\|\hat{x}\| = \|x\|$.

Finally, let $x_1, x_2 \in X$ and $x^* \in X^*$, then

$$\begin{aligned}(\widehat{x_1 + x_2})(x^*) &= x^*(x_1 + x_2)\\ &= x^*(x_1) + x^*(x_2)\\ &= \hat{x}_1(x^*) + \hat{x}_2(x^*).\end{aligned}$$

Similarly for any scalar $\alpha$ we have

$$\begin{aligned}(\widehat{\alpha x_1})(x^*) &= x^*(\alpha x_1)\\ &= \alpha x^*(x_1)\\ &= \alpha \hat{x}_1(x^*)\end{aligned}$$

Therefore the mapping $x \to \hat{x}$ is linear; and since $\|\hat{x}\| = \|x\|$, this mapping is Isometry.

That is, $x \to \hat{x}$ is a Linear Isometry of $X$ onto the linear sub-space $\{\hat{x} : x \in X^*\}$ of $X^{**}$.

**Definition 4.3.1.** Given a *NLS* $(X, \| \|)$, Linear Isometry $x \to \hat{x}$ is called the Canonical mapping of $X$ into its second conjugate space $X^{**}$.

**Definition 4.3.2.** A *NLS* ($X$, || ||) is called reflexive if and only if the Canonical mapping $x \to \hat{x}$ maps X onto X**.

Thus a necessary condition for $X$ to be reflexive is that $X$ is a Banach space. However there are Banach spaces without being reflexive.

## § 4.4 OPEN MAPPING THEOREM AND CLOSED GRAPH THEOREM :

Like a big and important theorem of Hahn-Banach we have another big theorem known as open mapping theorem in Functional analysis. There one is concerned with open mappings that send open sets into open sets. Open mapping theorem states conditions under which a bounded linear operator shall be an open mapping.

**Definition 4.4.1.** Let $X$ and $Y$ be two metric spaces. Then a mapping $f : X \to Y$ is called an open mapping if $G$ is an open set in $X$, its image under $f = f(G)$ is an open set in $Y$.

**Theorem 4.4.1.** Let ($X$, || ||) and ($Y$, || ||) be two Banach spaces; and $T : X \to Y$ be a bounded linear operator which is onto (surjective). Then $T$ is an open mapping.

The proof of the above theorem shall rest on following Lemma that we prove first.

**Lemma 4.4.1** Let $T : X \to Y$ be a bounded linear operator which is onto and let $B_0 = B_1(\underset{\sim}{0})$ be the open unit ball in $X$, then $T(B_0)$ contains an open ball centred at $\underset{\sim}{0}$ in $Y$.

**Proof :** We may complete the proof in three stages as under :

(a) $\overline{T(B_0)}$ (closure of $T(B_0)$) contains an open ball $B^*$.

(b) If $B_n$ = open ball $B_{\frac{1}{2^n}}(\underset{\sim}{0})$ in $X$, then $\overline{T(B_n)}$ shall contain an open ball $V_n$ centred at $\underset{\sim}{0}$ in $Y$.

and (c) $T(B_0)$ contains an open ball cenred at $\underset{\sim}{0}$ in $Y$.

(a) Consider open ball $B_1 = B_{\frac{1}{2}}(\underset{\sim}{0}) \subset X$. If $x \in X$, we find large real $k$ so that $x \in kB_1$. Therefore we write

$$X = \bigcup_{k=1}^{\infty} kB_1 \text{; Since } T \text{ is onto and linear, we have}$$

$$Y = T(X) = T\left(\bigcup_{k=1}^{\infty} kB_1\right) = \bigcup_{k=1}^{\infty} kT(B_1) = \bigcup_{k=1}^{\infty} \overline{kT(B_1)},$$ taking closure did not add more points to the Union = $Y$. As $Y$ is a Banach space, we invite Baire Category

Theorem to conclude that one component say $\overline{kT(B_1)}$ contains an open ball. That means $\overline{T(B_1)}$ contains an open ball, say, $B^* = B(y_0, \varepsilon)$. So we write

$$B^* - y_0 = B(\underset{\sim}{0}, \varepsilon) \subset \overline{T(B_1)} - y_0$$

(b) We show that $B^* - y_0 \subset \overline{T(B_0)}$, where $B_0$ stands as appearing in theorem. This is accomplished by showing :

$$\overline{T(B_1)} - y_0 \subset \overline{T(B_0)}$$

Take $y \in \overline{T(B_1)} - y_0$; then $(y + y_0) \in \overline{T(B_1)}$ and remembering that $y_0 \in \overline{T(B_1)}$ we find

$$u_n = T(w_n) \in T(B_1) \text{ such that } \lim_{n\to\infty} u_n = y + y_0$$

$$v_n = T(z_n) \in T(B_1) \text{ such that } \lim_{n\to\infty} v_n = y_0 .$$

Since $w_n, z_n \in B_1$ and $B_1$ is of radius = $\frac{1}{2}$ we have

$$\| w_n - z_n \| \le \| w_n \| + \| z_n \| < \frac{1}{2} + \frac{1}{1} = 1; \text{ So that } (w_n - z_n) \in B_0 .$$

From $T(w_n - z_n) - T(w_n) - T(z_n) = u_n - v_n \to y$ as $n \to \infty$.

Therefore, $y \in \overline{T(B_0)}$. Since $y \in \left(\overline{T(B_1)} - y_0\right)$ is an arbitrary we have shown

$$\overline{T(B_1)} - y_0 \subset \overline{T(B_0)}$$

From $B^* - y_0 = B(\underset{\sim}{0}, \varepsilon) \subset \overline{T(B_1)} - y_0$ above we have

$$B^* - y_0 = B(\underset{\sim}{0}, \varepsilon) \subset \overline{T(B_0)} \qquad \text{................ (1)}$$

Take $B_n = B(\underset{\sim}{0}, 2^{-n}) \subset X$. Since $T$ is linear, we have $\overline{T(B_n)} = 2^{-n}\overline{T(B_0)}$;
From (1) one obtains

$$V_n = B(\underset{\sim}{0}, \frac{\varepsilon}{2^n}) \subset \overline{T(B_n)} \qquad \text{.................. (2)}$$

(c) Finally, we show that $V_1 = B(\underset{\sim}{0}, \tfrac{1}{2}\varepsilon) \subset T(B_0)$.

Take $y \in V_1$. From (2), for $n = 1$, we have $V_1 \subset \overline{T(B_1)}$.

Hence $y \in \overline{T(B_1)}$ and we find $v \in \overline{T(B_1)}$ such that $\| y - v \| < \frac{\varepsilon}{4}$

Now $v \in \overline{T(B_1)}$ implies $v \in T(x_1)$ for some $x_1 \in B_1$.

Therefore $\| y - T(x_1) \| < \frac{\varepsilon}{4}$

Using this and (2) above with $n = 2$ we see that $\left(y - T(x_1)\right) \in V_2 \subset \overline{T(B_2)}$.

As before we find $x_2 \in B_2$ such that $\| y - T(x_1) \| - T(x_2) \| < \frac{\varepsilon}{8}$

Hence $(y - T(x_1) - T(x_2) \in V_3 \subset \overline{T(B_3)}$, and so on. In $n$th step we take $x_n \in B_n$ such that

$$\left\| y - \sum_{k=1}^{n} T(x_k) \right\| < \frac{\varepsilon}{2^{n+1}}, \quad n = 1, 2, \ldots \qquad \ldots\ldots\ldots\ldots\ldots (3)$$

Put $z_n = x_1 + x_2 + \ldots + x_n$; Since $x_k \in B_k$, we have $\| x_k \| < \frac{1}{2^k}$ that means $n > m$,

$$\| z_n - z_m \| \le \sum_{k=m+1}^{n} \| x_k \| < \sum_{k=m+1}^{\infty} \frac{1}{2^k} \text{ which } \to 0 \text{ as } m \to \infty.$$

So $\{z_n\}$ is Cauchy; let $\lim_{n \to \infty} z_n = x$ ($X$ is a Banach space).

Also $x \in B_0$ since $B_0$ has radius = 1, and

$$\sum_{k=1}^{\infty} \| x_k \| < \sum_{k=1}^{\infty} \frac{1}{2^k} = 1.$$

As $T$ is continuous, we have $\lim_{n \to \infty} T(z_n) = T(x)$ and (3) shows that $T(x) = y$. So $y \in T(B_0)$.

**Proof of Theroem 4.4.1.** If $A$ is an open set in $X$, we show that $T(A)$ is open in $Y$, by showing that every $y \in T(x) \in T(A)$ attracts an open ball centred at $y = T(x)$ within $T(A)$.

Take $y = T(x) \in T(A)$. As $A$ is open there is an open ball centred at $x \subset A$. Hence $A - x$ contains an open ball centred at $\underset{\sim}{0} \in X$. Let radius of that open ball = $r$. Put $k = \frac{1}{r}$ or $r = \frac{1}{k}$. Then $k(A - x)$ contains the open unit ball $B(\underset{\sim}{0}.1)$. Now Lemma 4.4.1 says that $T\left(k(A-x)\right) = k[T(A) - T(x)]$ contains an open ball centred at $\underset{\sim}{0}$, and so does $T(A) - T(x)$. Hence $T(A)$ contains an open ball centred at $y = T(x)$. As $y$ is an arbitrary member of $T(A)$, we have shown that $T(A)$ is open.

**Corollary : Under open mapping theorem if *T* is bijective, $T^{-1}$ is bounded.**

**Example 4.4.1.** Let $T : R^2 \to R$ be defined by $T(x,y) = x$ for $(x,y) \in R^2$. Show that $T$ is an open mapping. Examine if $T : R^2 \to R^2$ where $T(x, y) = (x, 0)$, $(x, y) \in R^2$ is an open mapping.

**Solution :** Here $T : R^2 \to R$ given by $T(x, y) = x$ is a projection mapping and we know that it is a bounded linear operator such that $T$ is onto. So we apply open mapping theorem to conclude that $T$ is an open mapping (In fact, $T$ sends open circular disc of $R^2$ onto an open interval).

If $T : R^2 \to R^2$ is given by $T(x, y) = (x, 0)$; there Image of an open circular disc under $T$ is not like that. So $T$ is not an open mapping.

We know that all linear operators are bounded. For instance, differential operator is an unbounded linear operator. Closed Linear operators that we introduce presently behave satisfactorily in this respect. Another important theorem, known as closed Graph Theorem states sufficient conditions under which a closed linear operator on a Banach space is bounded.

Let $(X \,||\;||)$ and $(Y, \,||\;||)$ be *NLS* with same scalars.

**Definition 4.4.2.** A linear operator $T : X \to Y$ is called a closed linear operator if its graph $G(T) = \{(x, y) \in (X \times Y) : y = T(x), x \in X\}$ is a closed set in *NLS* $X \times Y$ with norm $||(x, y)|| = ||x|| + ||y||$, $(x, y) \in (X \times Y)$.

**Theorem 4.4.2.** Let $X$ and $Y$ be Banach spaces, and $T : X \to Y$ be a closed linear operator. Then $T$ is a bounded linear operator.

**Proof :** First we verify that $X \times Y$ with norm $||(x, y)|| = ||x|| + ||y||$ as $(x, y) \in (X \times Y)$ is also a Banach space.

Let $\{z_n = (x_n, y_n)\}$ be a Cauchy sequence in $X \times Y$.

Then $||z_n - z_m|| = ||x_n - x_m|| + ||y_n - y_m||$

Thus $\| x_n - x_m \| \le \| z_n - z_m \| \to 0$ as $n, m \to \infty$ shows that $\{x_n\}$ is Cauchy in $X$, and since $X$ is complete,

let $\lim_{n\to\infty} x_n = x \in X$, and similarly let $\lim_{n\to\infty} y_n = y \in Y$.

These together imply that $\lim_{n\to\infty} z_n = z = (x, y) \in (X \times Y)$. Thus we see that $X \times Y$ is a Banach space. Graph $G(T)$ being a closed set in $X \times Y$, it follows that $G(T)$ is complete (infact, $G(T)$ is a Banach space as a subspace of $X \times Y$)

Consider a mapping $p : G(T) \to X$ given by $p(x, T(x)) = x \in X$. Then $p$ is linear operator over $G(T)$. $p$ is also bounded, because

$$\| p(x, T(x)) \| = \| x \| \le \| x \| + \| T(x) \| = (x, T(x)) \|$$

Further, $p$ is bijective; with $p^{-1}$ given by

$p^{-1} : X \to G(T)$ mapping $x \to (x, T(x))$ as $x \in X$. By applying open mapping theorem we find $p^{-1}$ to be bounded. Hence there is a +ve $K$ such that

$$\| (x, T(x)) \| \le K \| x \| \quad \text{for} \quad x \in X.$$

Therefore $\| T(x) \| \le \| T(x) \| + \| x \| = \| (x, T(x)) \| \le K \| x \|$.

That means $T$ is bounded. The proof is complete.

**Example 4.4.2.** If $X$ and $Y$ are Banach spaces over same scalars, and $T : X \to Y$ is a linear operator. Show that Graph $G(T)$ is a subspace of $X \times Y$.

**Solution :** Let $(x_1, T(x_1))$ and $(x_2, T(x_2))$ be two members of $G(T)$ as $x_1, x_2 \in X$, where $G(T) = \{(x, T(x)) : x \in X\} \subset (X \times Y)$.

Then $(x_1, T(x_1)) + (x_2, T(x_2)) = (x_1 + x_2, T(x_1) + T(x_2))$

$$= (x_1 + x_2, T(x_1 + x_2)) \quad (T \text{ is linear})$$

$$\in G(T).$$

If $\lambda$ is any scalar $\lambda(x_1, T(x_1)) = (\lambda x_1, \lambda T(x_1)) = (\lambda x_1, T(\lambda x_1)) \in G(T)$.

Therefore $G(T)$ is a sub-space of $(X \times Y)$.

## EXERCISE A

**Short answer type questions**

1. Show that a norm in a linear space $X$ is a sub-linear functional over $X$.
2. Show that a sub-linear functional $p$ in a linear space $X$ satisfies (a) $p(\underset{\sim}{0}) = 0$ and (b) $p(-x) \geq -p(x)$ for $x \in X$.
3. Show that non-null *NLS* $X$ has a non-null conjugate space $X^*$.
4. If $f(x) = f(y)$ for every bounded linear functional on a *NLS* $X$, show that $x = y$ in $X$.
5. If $X$ and $Y$ are Banach spaces show that the Null space $N(T)$ of a closed linear operator $T : X \to Y$ is a closed sub-space of $X$.
6. If two non-zero linear functionals $f_1$ and $f_2$ over a linear space have the same Null space, then show that $f_1$ and $f_2$ are proportional.

## EXERCISE B

1. Let $X$ be a *NLS*, and $x_0 \in X$ such that $| f(x_0) | \leq c$ for all $f \in X^*$ with $\| f \| = 1$, show that $\| x_0 \| \leq c$.
2. If $X$ is a *NLS* which is reflexive, show that $X^*$ is reflexive.
3. If $X$ and $Y$ are Banach spaces over the same scalars, and $T : X \to Y$ is a closed linear operator, then show that (a) if $C$ is compact in $X$, $T(C)$ is closed in $Y$, and (b) if $K$ is compact in $Y$, $T^{-1}(K)$ is closed in $X$.
4. Let $f$ be a non-zero linear functional in a linear space $X$, and $x_0$ is a fixed element

   in $\left(X \setminus N(f)\right)$, ($N(f)$ = Null space of $f = \{x \in X : f(x) = 0\}$), then any member

   $x$ in $X$ has a unique representation $x = \alpha x_0 + y$ where $y \in N(f)$. Prove it.
5. Show that $T : C[a,b] \to R$ defined by $T(f) = \int_a^b f dt, f \in C[a,b]$ is a bounded linear functional over $C[a,b]$ and find $\| T \|$.
6. Show that $f$ defined over $C[-1,1]$ by the rule :

   $$f(x) = \int_{-1}^{0} x dt - \int_{0}^{1} x dt, \quad x \in C[-1,1]$$

   is a bounded linear functional over $C[-1, 1]$ and find $\| f \|$.