UNIT 5

(***Contents*** : Inner product spaces, Cauchy-Schwarz inequality, I.P. spaces as *NLS*, continuity of I.P. function, Law of parallelogram, orthogonal (orthonormal) system of vectors, Projection Theorem in Hilbert space H; Reisz Theorem for a bounded linear functional over H, Bessel's inequality, Grahm-Smidst orthogonalisation process, complete orthonormal system in H.)

§ 5.1 INNER PRODUCT SPACE

In a Normed Linear space principle operations involved are addition of vectors and scalar multiplication of vectors by scalars as in elementary vector algebra. Norm in such a space generalizes elementary idea of length of a vector. What is still more missing in an *NLS* is an analogue if well known dot product $a.b = a_1b_1 + a_2b_2 + a_3b_3$, and resulting formulas among other things like (i) length $|a| = \sqrt{a.a}$ and (ii) relation of orthogonality $a.b = 0$. These are important tools in numerous applications.

History of Inner product spaces is older than that of *NLS*. Theory had been initiated by Hilbert through his work on integral equations. An inner product space is a Linear space with an inner-product structure that we presently define.

Suppose X denotes a complex Linear space.

Definition 5.1.1. X is said to be an Inner Product space or simply I.P. space if there is a scalar-valued function known Inner product function, denoted by, $<, >$ over $X_x X$ satisfying

(I.P. 1) $<x + y, z> = <x, z> + <y, z>$ for all $x, y, z \in X$,

(I.P. 2) $<\alpha x, y> = \alpha<x, y>$ for all scalars α and for all $x, y \in X$,

(I.P. 3) $<y, x> = \overline{<x, y>}$ for all $x, y \in X$, bar denoting complex conjugate.

(I.P. 4) $<x, x> \geq 0$ for all $x \in X$ and it is = 0 if and only if $x = \underline{0}$ in X.

From I.P. axioms above one can immediately derive the following :

(a) $<x, \alpha y> = \bar{\alpha} <x, y>$ for all scalars α and $x, y \in X$.

(b) $<\lambda x + \mu y, z> = \lambda <x, z> + \mu <y, z>$ for all $x, y, z \in X$ and for all scalars λ, μ.

(c) $<x, \alpha y + \beta z> = \overline{<\alpha x + \beta z, x>} = \overline{\alpha <y, x>} + \overline{\beta <z, x>}$

$= \overline{\alpha}\ \overline{< y, x >} + \overline{\beta}\ \overline{< z, y >} = \overline{\alpha} < x, y > + \overline{\beta} < x, z >,$ because conjugate of a complex scalar is itself.

Example 5.1.1. Unitary space $¢^n = \underbrace{¢ \times ¢ \times .... \times ¢}_{n\ copies}$ whose ¢ is the space of all complex number is an I.P. space with I.P. < > given by

$< \underset{\sim}{z}, \underset{\sim}{w}) = z_1\overline{w}_1 + z_2\overline{w}_2 + ... + z_n\overline{w}_n$ where $\underset{\sim}{z} = (z_1, z_2, ...z_n)$ and $\underline{w} = (w_1, w_2, ...w_n) \in ¢^n$.

Solution : Here $\overline{< \underset{\sim}{z}, \underset{\sim}{w} >} = \overline{z_1\overline{w}_1 + z_2\overline{w}_2 + ... + z_n\overline{w}_n}$

$= \overline{z_1\overline{w}_1} + \overline{z_2\overline{w}_2} + ... + \overline{z_n\overline{w}_n} = \overline{z}_1 w_1 + \overline{z}_2 w_2 + ... + \overline{z}_n w_n$

$= < \underset{\sim}{w}, \underset{\sim}{z} >$; and this (I.P. 3); rest of axioms are routine check-ups.

In an I.P. space (*X*, < >) of $x \in X$, let us define $\|x\|^2 = < x, x >$ which is always a non-negative quantity and is equal to 0 if and only if $x = \underline{0}$ in *X*.

Theorem 5.1.1. Every I.P. space is an *NLS*. To prove this Theorem we need help from following Lemma that is an independent proposition as well.

Lemma 5.1.1 (Cauchy-Schwarz inequality/C-S inequality)

In an I.P. space (*X*, < >) if *x*, *y* ∈ *X*,

$$|< x, y > \ \leq \ \|x\| \|y\|.$$

Proof : Without loss of generality take $y \neq \underset{\sim}{0}$ in *X*. (taking $y = \underset{\sim}{0}$ L.H.S. = R.H.S. For any scalar λ we have

$$\|x + \lambda y\|^2 \geq 0$$

or, $< x + \lambda y, x + \lambda y > = 0$

or, $< x, y > + \lambda\overline{\lambda} < y, y > + \overline{\lambda} < x, y > + \overline{\lambda} < y, x > \geq 0$

or, $\|x\|^2 + |\lambda|^2 \|y\|^2 + \overline{\lambda} < x, y > + \lambda\overline{< x, y >} \geq 0$

Let us now choose $\lambda = -\dfrac{< x, y >}{< y, y >}$

$$= -\frac{< x, y >}{\|y\|^2}.$$

Then L.H.S. of above inequality

$$= \| x \|^2 + \frac{| x, y |^2}{\| y \|^2} - \frac{|< x, y >|^2}{\| y \|^2} - \frac{|< x, y >|^2}{\| y \|^2} = \| x \|^2 - \frac{|< x, y >|^2}{\| y \|^2}$$

Therefore above inequality assumes the form

$$\| x \|^2 - \frac{|< x, y >|^2}{\| y \|^2} \geq 0$$

or $\qquad |< x, y > \leq \| x \| \| y \|.$

Proof of Theorem 5.1.1. Norm axioms (N.1) and (N.2) follow from (I.P. 4); and the fact $\| \alpha x \|^2 = < \alpha x, \alpha x > = \alpha \bar{\alpha} < x, x > = | \alpha |^2 \| x \|^2$.

This gives $\| \alpha x \|^2 = | \alpha | \| x \|$

For triangle inequality (N.3), let *x, y* ∈ *X,* then we have

$$\| x + y \|^2 = < x + y, x + y > = \| x \|^2 + < x, y > + < y, x > + \| y \|^2.$$

Thus $\| x + y \|^2 \leq \| x \|^2 + |< x, y > + |< y, x >| + \| y \|^2$

$$= \| x \|^2 + 2 |< x, y >| + \| y \|^2$$

$$\leq \| x \|^2 + 2 \| x \| \| y \| + \| y \|^2 \text{ by Lemma 5.1.1.}$$

$$= (\| x \| + \| y \|)^2.$$

Therefore $\| x + y \| \leq \| x \| + \| y \|$.

The proof is now complete.

Remark : Equality sign in C-S inequality holds if and only if $y = 0$ or $0 = \| x + \lambda y \|^2$ *i.e.* $x = -\lambda y$ or $x + \lambda y = 0$ showing that *x* and *y* to be linearly dependent.

Theorem 5.1.2. In an I.P. space (*X*, < >), show that I.P. function is a continuous function.

Proof : Let $\{x_n\}$ and $\{y_n\}$ be two sequences in *X* such that $\lim_n z_n = x$ and $\lim_{x \to \infty} y_n = y$ in norm. That is to say, $\lim_{n \to \infty} \| x_n - x \| = 0 = \lim_{n \to \infty} \| y_n - y \|$.

Now $|< x_n, y_n > - < x, y >| = |< x_n, y_n > - < x_n, y > + < x_n, y > - < x, y >|$

$$= |< x_n, y_n - y > + < x_n - x, y >|$$

$$\leq |< x_n, y_n - y >| + |< x_n - x, y >|$$

$$\leq || x_n || || y_n - y || + || y || || x_n - x ||;$$

Since $\lim_{x \to \infty} x_n = x$ in norm, $\{x_n\}$ is norm bounded; So there is an M (+ve) such that $|| x_n || \leq M$ for all n.

Therefore above inequality assumes the form

$\leq M || y_n - y || + || y || || x_n - x || \to 0$ as $n \to \infty$. This shows that $\lim_{n \to \infty} < x_n, y_n > = < x, y >$ and I.P. function is continuous at (x, y).

Definition 5.1.2. An I.P. space X is said to be a Hilbert space if X is a complete *NLS* with norm $|| \; ||$ as induced from I.P. function.

Thus every Hilbert space is a Banach space. But opposite is not true.

Very often a Hilbert space is denoted by H and an I.P. space is termed as a pre-Hilbert space.

Theorem 5.1.3. If x and y are two members in a Hilbert space H, then

$|| x + y ||^2 + || x - y ||^2 = 2 || x ||^2 + 2 || y ||^2$. (Law of parallelogram).

Proof : Here $|| x + y ||^2 + || x - y ||^2 = < x + y, x + y > + < x - y, x - y >$

$$= || x ||^2 + < x, y > + < y, x > + || y ||^2 + || x ||^2 - < x, y > - < y, x > + || y ||^2$$

$$= 2 || x ||^2 + 2 || y ||^2.$$

Remark : In school Geometry it is known that sum of squares raised on sides of a parallelogram is equal to the sum of squares raised on its diagonals. This is exactly what is in Theorem 5.1.3 above. Hence the name is Law of parallelogram.

Example 5.1.2. The sequence space l_2 of all real sequences $\underset{\sim}{x} = \{\xi_1, \xi_2, ..., \xi_n, ...\}$ with $\sum_{i=1}^{\infty} |\xi_i|^2 < \infty$ is a real Hilbert space.

Solution : We know that l_2 is a real linear space where let us define an I.P. function $< \underset{\sim}{x}, \underset{\sim}{y} > = \sum_{i=1}^{\infty} \xi_i \eta_i$, the r.h.s. series is convergent because

$|\xi_i \eta_i| \leq \frac{1}{2}(|\xi_i|^2 + |\eta_i|^2 \; [(\underset{\sim}{x} = (\xi_1, \xi_2, ...), \; \underset{\sim}{y} = (\eta_1, \eta_2, ...) \in l_2]$. $\quad i = 1, 2,.........$

By routine exercise we check that all I.P. axioms are O.K. in l_2, and l_2 is an I.P.

space with real scalars. Further, with respect to the induced norm $\| \underset{\sim}{x} \|^2 = \sum_{i=1}^{\infty} |\xi_i|^2$ it is also known that l_2 becomes a complete *NLS*. Hence l_2 is a Hilbert space.

Example 5.1.3. The sequence space $l_p (1 < p < \infty)$ consisting of all real sequences $\underset{\sim}{x} = (\xi_1, \xi_2, ...)$ with $\left(\sum_{i=1}^{\infty} |\xi_i| p \right)^{1/p} < \infty$ is a Banach space without being a Hilbert space with I.P. function to induce Banach-space norm.

Solution : We have already seen that sequence space $l_p (1 < p > \infty)$ is a Banach space with norm $\| x \| = \left(\sum_{i=1}^{\infty} |\xi_i| p \right)^{1/p}$, as $\underset{\sim}{x} = (\xi_1, \xi_2, ...) \in l_p$. We now show that this norm does not come from an I.P. function on l_p. This is verified by showing that this norm does not satisfy Law of Parallelogram. Take $\underset{\sim}{x} = \left(1, 1, \frac{0}{\text{block}}\right)$, $\underset{\sim}{y} = \left(1, -1, \frac{0}{\text{block}}\right)$ from l_p. Then we find $\| \underset{\sim}{x} \| = \| \underset{\sim}{y} \| = 2^{1/p}$ and $\| \underset{\sim}{x} + \underset{\sim}{y} \| = 2$ $= \| \underset{\sim}{x} - \underset{\sim}{y} \|$. Therefore, if $p \neq 2$ parallelogram law fails.

§ 5.2 ORTHOGONAL ELEMENTS IN HILBERT SPACE

Let *H* denote a Hilbert space.

Definition 5.2.1. (a) Two members *x* and *y* in a Hilbert space H are called orthogonal if $< x, y > = 0$;

We write in this case $x \perp y$.

(b) Given a non-empty subset *L* of *H*, an element $x \in$ H is said to be orthogonal to *L*, denoted by $x \perp y$ if $< x, l > = 0$ for every member $l \in L$.

Theorem 5.2.1. (Pythogorian Law) If $x, y \in H$ and $x \perp y$, then

(i) $\| x + y \|^2 = \| x \|^2 + \| y \|^2$

(ii) $\| x - y \|^2 = \| x \|^2 + \| y \|^2$

Proof : (i) $\| x + y \|^2 = < x + y, x + y > = \| x \|^2 + < x, y > + < y, x > + \| y \|^2$

$= \| x \|^2 + < x, y > + \overline{< x, y >} + \| y \|^2 = \| x \|^2 + \| y \|^2$ since $<x, y> = 0$.

(ii) the proof is similar to above.

Theorem 5.2.2. Every closed convex subset of a Hilbert space H has a unique member of smallest norm.

Proof : Let C be a closed convex subset of H, and let d = Inf $\{\| x \| : x \in C\}$.

Let $\{x_n\}$ be a sequence in C such that $\lim_{n\to\infty} \| x_n \| = d$. for $x_n, x_m \in C$ we have $\frac{1}{2}(x_n + x_m) \in C$, because C is convex.

So, $\left\| \frac{x_n + x_m}{2} \right\| \geq d$ or, $\| x_n + x_m \| \geq 2d$ .... (1)

By Law of Parallelogram we have

$$\| x_n - x_n \|^2 = 2 \| x_n \|^2 + 2 \| x_m \|^2 - \| x_n + x_m \|^2$$

$$\leq 2 \| x_n \|^2 + 2 \| x_m \|^2 - 4d^2.$$

Since $\lim_{n\to\infty} \| x_n \| = d$ and similarly $\| x_m \| \to d$ as .... (2)

$m \to \infty$; taking limit n, $m \to \infty$ in (2) we get

$\lim_{n,m\to\infty} \| x_n - x_m \| = 0$; showing that $\{x_n\}$ is Cauchy in C.

As C is closed, Let $\lim_{n\to\infty} x_n = x \in C$. Thus $\| x \| = \lim_{n\to\infty} \| x_n \| = d$.

Hence $x \in C$ has a smallest norm. For uniqueness of x, let $x' \in C$ so that $\| x' \| = d$. By convexity of C we have $\frac{x + x'}{2} \in C$ and also $\| \frac{x + x'}{2} \| \geq d$. Again by Law of Parallelogram we have

$$\| \frac{x + x'}{2} \|^2 = \frac{\| x \|^2}{2} + \frac{\| x' \|^2}{2} - \frac{\| x - x' \|^2}{2}$$

$$< \frac{1}{2} \| x \|^2 + \frac{1}{2} \| x' \|^2 \quad \text{if } x \neq x'$$

$= d^2$; giving $\| \frac{x + x'}{2} \| < d$—a contradiction of $\| \frac{x + x'}{2} \| \geq d$ as arrived at early. The proof is now complete.

Theorem 5.2.3 (Projection Theorem). Let L be a closed subspace of H and $L \neq H$; Then every member $x \in H$ has a unique representation $x = y + z$ where $y \in L$ and $x \perp L$.

Proof : If x is a member of $L \subset H$; we write $x = x + \underset{\sim}{0}$ when $\underset{\sim}{0} \in \perp L$.

Let us take $x \in (H \setminus L)$, and put

$d = \operatorname*{Inf}_{a \in L} \| x - a \|^2 = dist(x, L)$; Because L is closed we have $d > 0$,

and there is a sequence $\{a_n\}$ of member a_n in L such that

$$\lim_{n \to \infty} d_n = \| x - a_n \|^2 = d. \qquad \text{............... (1)}$$

Take any non-zero member a in L. As L is a sub-space of H, we have for any scalar ε, $(a_n + \varepsilon a) \in L$ and therefore

$$\| x - (a_n + \varepsilon a) \|^2 \geq d$$

or, $\quad < x - a_n - \varepsilon a,\ x - a_n - \varepsilon a > \geq d$

or, $\quad \| x - a_n |^2 - \bar{\varepsilon} < x - a_n, a > - \varepsilon < a, x - a_n > + | \varepsilon |^2 \| a \|^2 \geq d$.

Now take $\varepsilon = \dfrac{< x - a_n, a >}{\| a \|^2}$; with such a choice of ε, we have

$$\| x - a_n \|^2 - \frac{|< x - a_n, a >|^2}{\| a \|^2} \geq d$$

or, $\quad |< x - a_n, a >|^2 \leq \| a \|^2 \ (d_n - d)$

or, $\quad |< x - a_n, a >| \leq \| a \| \sqrt{d_n - d} \qquad$ .... (*)

Inequality holds for $a = \underset{\sim}{0}$ in L; So for any $a \in L$ we have

$$|< a_n - a_m, a >| \leq |< a_n - x, a >| + |< x - a_m, a >|$$

i.e. $\quad |< a_n - a_m, a >| \leq \| a \| \left(\sqrt{d_n - d} + \sqrt{d_m - d} \right) \qquad$ from (*)

Putting $a = a_n - a_m$, we have

$$|< a_n - a_m, a_n - a_m >| \leq \| a_n - a_m \| \left(\sqrt{d_n - d} + \sqrt{d_m - d} \right)$$

i.e. $\quad \| a_n - a_m \|^2 \leq \| a_n - a_m \| \left(\sqrt{d_n - d} + \sqrt{d_m - d} \right)$

or, $\| a_n - a_m \| \le \left(\sqrt{d_n - d} + \sqrt{d_m - d}\right)$, where r.h.s. $\to 0$ as $n, m \to \infty$ by (1).

That means $\{a_n\}$ is Cauchy in L.

Since L is closed, let $\lim_{n\to\infty} a_n = y \in L$.

Now in $|< x - a_n, a >| \le \| a \| \sqrt{d_n - d}$, let us pass on the $\lim_{n\to\infty} a_n = y$ and get $|< x - y, a >| = 0$

i.e. $< x - y, 0 > = 0$; This is true for any member a in L; Therefore $(x - y) \perp L$. Let us put $z = x - y$.

Then we have $x = y + z$ where $y \in L$ and $z \perp L$.

For uniqueness of this representation, let $x = y + z = y' + z'$ where $y' \in L$ and $z' \perp L$. Thus y, y' come from L and $z, z' \perp L$. Clearly, $y - y' = z' - z$, and

$\| y - y' \|^2 = < y - y', y - y' > = < y - y', z' - z > = 0$ where $\| z' - z \| \perp L$.
Therefore $y = y'$ and hence $z = z'$. The proof is now complete.

Remark : In representation Theorem 5.2.3. where $x = y + z$, y is called projection of x on L. It is obvious that collection M of all elements, orthogonal to L forms a sub-space. M is also closed because of continuity of I.P. function. That is why z is called projection of x on M which is called orthogonal complement of L. Further, Hilbert space H is then sum of two orthogonal sub-spaces L and M. Here we see orthogonal sum is a special case of the Direct sum. Thus projection Theorem 5.2.3 gives a decomposition of any member in Hilbert space H into its projections onto two complementary orthogonal sub-spaces.

§ 5.3. It is important to know that the general form of a bounded Linear functional acting on a given space. Such formulae in respect of some *NLS* are known; their derivations could be much complicated. Situation is, however, surprisingly simple for a Hilbert space H.

Theorem 5.3.1 (Riesz Theorem on representation of functional over H).

Let f be a bounded linear functional over a Hilbert space H. Then $f(x) = < x, y >$ for all $x \in H$ and for some $z \in H$ uniquely determined by f such that $\| z \| = \| f \|$.

Proof : If f is the zero functional over H. We take $z = \underset{\sim}{0}$ in H to do the job. Suppose that f is a non-zero bounded linear functional over H. Consider the null-space $N(f)$ of f where

$N(f) = \{x \in H : f(x) = 0\}$. Clearly $N(f)$ is a closed linear sub-space of H without being equal to H.

Take a non-zero $z_0 \in \perp N(f)$

Let $x \in$ H. Put $v = f(x)z_0 - f(z_0)x$

So that $f(v) = f(f(x)z_0) - f(f(z_0)x)$

$= f(x)f(z_0) - f(z_0)f(x)$; (f is linear)

$= 0$

That means $v \in N(f)$; by choice z_0 is orthogonal to v

So $0 = < v, z_0 > = < f(x)z_0 - f(z_0)x, z_0 >$

$= f(x) < z_0, z_0 > - f(z_0) < x, z_0 >$

$= \|z_0\|^2 f(x) - f(z_0) < x, z_0 >$

Giving $f(x) = \dfrac{f(z_0)}{\|z_0\|} < x, z_0 >$

$$= \left\langle x, \ \overline{\frac{f(z_0)}{\|z_0\|}} z_0 \right\rangle$$

$= < x, y >$ (say), where $z = \dfrac{\overline{f(z_0)}}{\|z_0\|^2} z_0$. .... (1)

This is the representative formula for $f(x)$ as wanted.

For uniqueness of z, let $f(x) =< x, z_1 > = < x, x_2 >$ for all $x \in \mathrm{H}$.

Then we have $< x, z_1 > = < x, z_2 >$ or, $< x, z_1 - z_2 > = 0$

put $x = z_1 - z_2$; So $< z_1 - z_2, \ z_1 - z_2 > = 0$ or, $\|z_1 - z_2\|^2 = 0$ or, $z_1 = z_2$.

Finally, We have $|f(x)| = |< x, z >| \leq \|x\| \|z\|$

This gives $\|f\| \leq \|z\|$ .... (1)

Again taking $z = x$ in (1) we have $< z, z > = f(z)$

or, $\|z\|^2 \leq \|f\| \|z\|$

or, $\|z\| \leq \|f\|$ .... (2)

Combining (1) and (2) we have $\|f\| \leq \|z\|$.

Converse of Theorem 5.3.1. is true. This is what Example 5.3.1 has to say.

Example 5.3.1. Let *z* be a fixed member in a Hilbert space *H*. Show that

$f(x) = <x, z>$ for all $x \in$ H is a bounded linear functional over *H* with $\|f\| = \|z\|$.

Solution : Here *f* : *H* →Scalar such that for $x_1, x_2 \in H$.

Then $f(x_1 + x_2) = <x_1 + x_2, z> = <x_1, z> + <x_2, z> = f(x_1) + f(x_2)$.

And for any scalar α $f(\alpha x_1) = <\alpha x_1, z> = \alpha <x_1, z> = \alpha f(x_1)$.

Thus *f* is Linear. Further $|f(x)| = |<x, z>| \leq \|x\| \|z\|$ (by C-S inequality)

This is true for all $x \in$ H. Therefore *f* is a bounded linear functional such that

$$\|f\| \leq \|z\| \qquad \text{............... (1)}$$

Taking *x* = *z* in *f*(*x*) = <*x*, *y*> we have

$$\|z\|^2 = <z, z> = f(z) \leq \|f\| \|z\|$$

$$\text{or, } \|z\| \leq \|f\| \qquad \text{............... (2)}$$

(1) plus (2) gives $\|f\| \leq \|z\|$.

Corollary to Theorem 5.3.1. Every Hilbert space H is reflexive.

Because by Theorem 5.3.1. together example put up above says that every bounded linear functional over *H*. *i.e.* every member of *H** arises out of a member of *H* and conversely. This correspondence gives rise to an isomorphism between *H* and *H**; and we say that *H* is self-dual and this in turn implies that here Canonical mapping between *H* and *H*** is a surjection. Hence *H* is reflexive.

§ 5.4 ORTHONORMAL SYSTEM IN HILBERT SPACE *H*.

Definition 5.4.1. (a) A non-empty subset $\{e_i\}$ of Hilbert space *H* is said to be an orthonormal system if

(i) $i \neq j$, $e_i + e_j$ *i.e.* any two distinct members of $\{e_i\}$ are orthogonal.

and (ii) $\|e_i\| = 1$ for every i *i.e.* any vector of the system is non-zero unit vector in *H*.

(b) If an orthonormal system of *H* is countable, we can enumerate its elements in a sequence say it as an orthonormal sequence.

For example in Euclidean n-space R^n which is a real Hilbert space the fundamental unit vectors $e_1 = (1,0,0,...,0)$, $e_2 = (0,1,0,0,...,0)$ ...... $e_n = (1,0,...,0,1)$ form an orthonormal system of vectors in R^n.

Example 5.4.1. Let $L_2[0,2\pi]$ be the real Hilbert space of all square integrable functions f over $[0,2\pi]$ with I.P. function

$$< f,g > = \int_0^{2\pi} fgdt; \qquad f,g \in L_2 \ [0,2\pi].$$

$$\therefore \| f \| = \sqrt{\sum_0^{2\pi} f^2 dt}.$$

Then $e_0(t) = \dfrac{1}{\sqrt{2\pi}}, e_n(t) = \dfrac{\cos nt}{\sqrt{\pi}}$ {n =1, 2,....) and $0 \le t \le 2\pi$.

form an orthonormal sequence in $L_2[0,2\pi]$; because

$$\int_0^{2\pi} \cos mt \ \cos nt \ dt = \begin{cases} 0 & \text{if } m \ne n \\ \pi & \text{if } m = n = 1,2... \\ 2\pi & \text{if } m = n = 0 \end{cases}$$

Theorem 5.4.1. An orthonormal system in H is linearly independent.

Proof : Let $\{e_i\}$ be an orthonormal system in H; and let for a finite subset, say, e_1, e_2, ...e_n of the system we have

$\alpha_1 e_1 + \alpha_2 e_2 + ... + \alpha_n e_n = \underset{\sim}{0}$ where α_i 's are scalars. Then for $1 \le j \le n$ we have

$$0 = < \underset{\sim}{0}, e_j > = \left\langle \sum_{i=1}^{n} \alpha_i e_i, e_j \right\rangle = \sum_{i=1}^{n} \alpha_i, < e_i, e_j >$$

$= \alpha_j < e_j, e_j > = \alpha_j$; (other terms being zero because of mutual orthogonality). So $\alpha_1 = \alpha_2 = ... = \alpha_n = 0$. That means any finite sub-system of the given system is linearly independent. Hence proof is done.

Definition 5.4.2. Let $\{e_i\}$ be an orthonormal system in H and $x \in H$; Then scalars $c_i =< x, e_i >$ are called Fouries co-efficients of x w.r.t the system.

Theorem 5.4.2. Suppose $\{e_1, e_2, e_3, ..., e_n, ...\}$ be an orthonormal sequence in H;

then for $x \in$ H,

$$\sum_{i=1}^{\infty} |< x, e_i >|^2 \le \| x \|^2$$

(This inequality is very often termed as Bessel's inequality).

Proof : Let n be a +ve integer. If c_i are Fouries coefficients of x w.r.t. $\{e_i\}$, we have

$$0 \le \| x - \sum_{i=1}^{n} c_i e_i \|^2 = \left\langle x - \sum_{i=1}^{n} c_i e_i ,\, x - \sum_{i=1}^{n} c_i e_i \right\rangle$$

$$= \| x \|^2 - \left\langle x, \sum_{i=1}^{n} c_i e_i \right\rangle - \left\langle \sum_{i=1}^{n} c_i e_i , x \right\rangle + \left\langle \sum_{i=1}^{n} c_i e_i , \sum_{k=1}^{n} c_k e_k \right\rangle$$

$$= \| x \|^2 - \sum_{i=1}^{n} \overline{c_i} < x, e_i > - \sum_{i=1}^{n} c_i < e_i , x > + \sum_{i=1}^{n} c_i < e_i , \sum_{k=1}^{n} c_k e_k >$$

$$= \| x \|^2 - \sum_{i=1}^{n} \overline{c_i} c_i - \sum_{i=1}^{n} c_i \overline{c_i} + \sum_{i=1}^{n} \sum_{k=1}^{n} c_i \overline{c_k} < e_i , e_k >$$

$$= \| x \|^2 - \sum_{i=1}^{n} | c_i |^2 - \sum_{i=1}^{n} | c_i |^2 + \sum_{i=1}^{n} | c_i |^2 = \| x \|^2 - \sum_{i=1}^{n} | c_i |^2$$

Therefore, $\sum_{i=1}^{n} | c_i |^2 \le \| x \|^2$ or, $\sum_{i=1}^{n} |< x, e_i >|^2 \le \| x \|^2$.

This is true for any +ve integer n, and thus $\sum_{i=1}^{\infty} |< x, c_i >|^2$ is convergent and $\sum_{i=1}^{n} |< x, c_i >|^2 \le \| x \|^2$.

Theorem 5.4.3. In a separable Hilbert space H every orthonormal system is countable.

Proof : Let $E = \{e_i\}$ be an orthonormal system in H which is separable. If $e_i \ne e_j$ we have $< e_i , e_j > = 0$ and $\| e_i \| = 1 = \| e_j \|$.

Therefore $<e_i - e_j, e_i - e_j> = \| e_i \|^2 - <e_i, e_j> - <e_j, e_i> + \| e_j \|^2 = 0+1+1=2$

So, $\| e_i - e_j \|^2 = 2$

or, $\| e_i - e_j \| = \sqrt{2}$.

By separability of H, we find a countable set $\{y_1, y_2, ..., y_n, ...\}$ which is dense in E. So we find two members, say, y_i and y_i such that

$$\| y_i - e_i \| < \frac{\sqrt{2}}{3} \text{ and } \| y_j - e_j \| < \frac{\sqrt{2}}{3}.$$

So $$\sqrt{2} = \| e_i - e_j \| = \| e_i - y_i + y_i - y_j + y_j - e_j \|$$

$$\le \| e_i - y_i \| + \| y_i - y_j \| + \| y_j - e_j \|$$

$$< \frac{2\sqrt{2}}{3} + \| y_i - y_j \|.$$

Showing $\| y_i - y_j \| > \frac{\sqrt{2}}{3}$. clearly $i \neq j$; This establishes an H correspondence between members of E with members of a subset of a countable set. Therefore E is countable.

Gram-Schmidt Orthogolisation Process : Subject is that in a Hilbert space H one can transform a linearly independent set of elements in H into an orthonormal system in H by a technique known by above name.

Let x_1, x_2,... be an independent system of vectors in H (So none is zero vector)

Put $e_1 = \frac{x_1}{\| x_1 \|}$ and let $y_2 = x_2 - c_{21} e_1$ where $c_{21} = <x_2, e_1>$.

Next we put $e_2 = \frac{y_2}{\| y_2 \|}$; By verification we see $<e_1, e_1> = 1$, $<e_2, e_2> = 1$, and $<e_1, e_2> = 0$.

Now let $y_3 = x_3 - (c_{31} e_1 + c_{32} e_2)$ where we choose $c_{31} = <x_3, e_1>$, $c_{32} = <x_3, e_2>$.

Next we put $e_3 = \frac{y_3}{\| y_3 \|}$, and as before we have

$$<e_3, e_3> = 1, \ <e_3, e_2> = 0 = <e_3, e_1>.$$

We continue this process, if $e_1, e_2, ..., e_{k-1}$ have been constructed, let us take

$$y_k = x_k - \sum_{i=1}^{k-1} c_{ki} e_i$$

where $c_{ki} = < x_k, e_i >$; so that y_k is orthogonal to $e_1, e_2, ..., e_{k-1}$; Define $e_k = \frac{y_k}{\| y_k \|}$. Inductively, we construct e_n as a linear combination of x_1, x_2,... and x_n. This way we are led to orthonormal system (e_1, e_2,... e_n,...) from (x_1, x_2,... x_n,...).

Definition 5.4.3. In a Hilbert space H an orthonormal system E is called a complete orthonormal system if there is no orthonormal system in H to contain E as a proper subset.

For example, in Euclidean n-space R^n (a real Hilbert space) the set of all fundamental unit vectors $\{e_1, e_2, ..., e_n\}$ where $e_j = (\underline{0} \underset{j\text{th place}}{\underline{1}} \underline{0})$, $j = 1, 2, ..., n$ is a complete orthonormal system in R^n.

Theorem 5.4.4. In a Hilbert space H let $\{e_1, e_2, ..., e_n ...\}$ be an orthonormal sequence in H. Then following statements are equivalent (one implies other).

(a) $\{e_i\}$ is complete.

(b) $< x, e_i > = 0$ for all i implies $x = \underset{\sim}{0}$ in H.

(c) $x = \sum_{i=1}^{\infty} < x, e_i > e_i$ for each $x \in H$.

(d) $\sum_{i=1}^{\infty} |< x, e_i >|^2 = \| x \|^2$ for every $x \in H$.

Proof : (a) $\Rightarrow$ (b); Let (a) be true. Suppose (b) is false. Then we find a non-zero x in H such that $< x, e_i > = 0$ for i = 1,2,...

Put $e = \frac{x}{\| x \|}$. So that $\| e \|$ = 1, and $< e, e_i > = 0$ for all i,

Therefore $\{e_1, e_2, ..., ... e_n, ...\} \cup \{e\}$ becomes an orthonormal system containing given system properly—a contradiction that $\{e_1, e_2, ..., e_n\}$ is complete. Hence (b) is established.

(b) $\Rightarrow$ (c) Let $S_n = \sum_{i=1}^{n} < x, e_i > e_i$;

Then $\sum_{i=1}^{\infty} < x, e_i > e_i = \lim_{n\to\infty} S_n = S$ (say)

If $1 \le j \le n$, $< x, e_j > - < S_n, e_j >$

$$= < x, e_j > - < \sum_{i=1}^{n} < x, e_i > e_i, e_j >$$

$$= < x, e_j > - < x, e_j > = 0$$

Thus $< S_n, e_j > = < x, e_j >$

Now $< x - \sum_{i=1}^{n} < x, e_i > e_i, e_j > = < x - S, e_j > = < x, e_j > - < S, e_j >$

$$= < x, e_j > - < \lim_{n\to\infty} S_n, e_j > = < x, e_j > - \lim_{n\to\infty} < S_n, e_j > = < x, e_j > - < x, e_j > = 0$$

That means $e_j \perp \left(x - \sum_{i=1}^{\infty} < x, e_i > e_i \right)$; therefore from (b) we have

$$x - \sum_{i=1}^{n} < x, e_i > e_i = \underset{\sim}{0} \quad i.e. \quad x = \sum_{i=1}^{\infty} < x, e_i > e_i .$$

<u>(c) $\Rightarrow$ (d)</u>. We have $\| x \|^2 = < x, x > = < \sum_{i=1}^{\infty} < x, e_i > e_i . \sum_{j=1}^{\infty} < x, e_j > e_j >$

$$= < \lim_{n\to\infty} \sum_{i=1}^{n} < x, e_i > e_i , < \lim_{n\to\infty} \sum_{j=1}^{n} < x, e_j > e_j >$$

$$= \lim_{n\to\infty} < \sum_{i=1}^{n} < x, e_i > e_i , \sum_{j=1}^{n} < x, e_j > e_j >$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} < x, e_i > \overline{< x, e_i >} = \lim_{n\to\infty} \sum_{i=1}^{n} |< x, e_i >|^2$$

$$= \sum_{i=1}^{\infty} |< x, e_i >|^2 .$$

$\underline{(d) \Rightarrow (a)}$. Let (*d*) hold and if possible let $\{e_i\}$ be not complete. Then we find an orthonormal system strictly larger than $\{e_1, e_2, ..., e_n, ...\}$; say larger system looks as $\{e, e_1, e_2, ..., e_n ....\}$ where, of course, $\| e \| = 1$ and $<e, e_i> = 0$ for $i = 1, 2, ...$ Now (d) applies (taking $x = e$), and we have

$$\| e \|^2 = \sum_{i=1}^{\infty} |< e, e_i >|^2 = 0 \text{— a contradiction. So we have proved (a).}$$

Example 5.4.2. Let $\{x_n\}$ be a sequence in Hilbert space H and $x \in H$ such that $\lim_{n\to\infty} \| x_n \| = \| x \|$, and $\lim_{n\to\infty} < x_n, x >\| = < x, x >$. Show that $\lim_{n\to\infty} x_n = x$.

Solution : Given $\lim_{n\to\infty} \| x_n \| = \| x \|$ and $\lim_{n\to\infty} < x_n, x > = < x, x > = \| x \|^2$.

Now $\| x_n - x \|^2 = < x_n - x, x_n - x > = \| x_n \|^2 - < x_n, x > - < x, x_n > + \| x \|^2$

$$= \| x_n \|^2 - < x_n, x > - \overline{< x_n, x >} + \| x \|^2$$

$$\to \| x \|^2 - \| x \|^2 - \| x \|^2 + \| x \|^2 = 0 \text{ as } n \to \infty.$$

Therefore $\lim_{n\to\infty} x_n = x$

Example 5.4.3. In a real Hilbert space H if $\| x \| = \| y \|$, show that $<x + y, x - y> = 0$. Interpret the result Geometrically if H = Euclidean 2-space R^2.

Solution : Let H be a real Hilbert space and $x, y \in H$ that such $\| x \| = \| y \|$.

Now $< x + y, x - y > = < x, x > - < x, y > + < y, x > - < y, y >$

$= \| x \|^2 - < x, y > + < x, y > - \| y \|^2$ (because it is a real Hilbert space, $< x, y > = \overline{< x, y >}$)

$= 0$

That means $(x + y) \perp (x - y)$.

It Euclidean 2-space R^2, fig is an equilateral parallelogram *i.e.* a Rhombus with adjacent sides represented by x and y with $\| x \| = \| y \|$; and we know that in a Rhombus Diagonals cut at right angles.

EXERCISE A

Short answer type questions

1. If in an I.P. space $< x, u > = < x, v >$ for all x in the space, show that $u = v$.

2. Show that Banach space $c[a,b]$ with sup norm is not a Hilbert space with an I.P. to induce the sup norm.
3. If f is a bounded linear functional over Euclidean 3-space R^3, show that f can be represented by a dot product

$$f(x) = x.z = \xi_1\rho_1 + \xi_2\rho_2 + \xi_3\rho_3 .$$

4. Show that in a Hilbert space H convergence of $\sum_{j=1}^{\infty} \| x_j \|$ implies convergence of $\sum_{j=1}^{\infty} x_j$
5. If ¢ denotes the Unitary space of all complex numbers. If z_1, z_2 ∈ ¢, show that $< z_1, z_2 > = z_1, \overline{z}_2$ defines an I.P. function on ¢.

EXERCISE B

1. If x and y are two non-zero elements in a Hilbert space H, show that $\| x + y \| \leq \| x \| + \| y \|$ where equality holds if and only if $y = \alpha x$ for a suitable scalar α.
2. Let c be a convex set in a Hilbert space H, and $d = \text{Inf}\{\| x \| : x \in c\}$. If $\{x_n\}$ is a sequence in c such that $\lim_{n \to \infty} \| x_n \| = d$, show that $\{x_n\}$ is a Cauchy sequence.
3. If $\{e_n\}$ is any orthonormal sequence in a Hilbert space H and x, $y \in H$, show that

$$\left| \sum_{n=1}^{\infty} < x, e_n > < y, e_n > \right| \leq \| x \| \| y \|$$

4. Let $\{e_1, e_2, ..., e_n\}$ be an orthonormal set in a Hilbert space H where n is fixed. If $x \in H$ be a fixed member, show that for scalars $\alpha_1, \alpha_2, ..., \alpha_n$ $\| x - \sum_{i=1}^{n} \alpha_i e_i \|$ is minimum when $\alpha_i = < x, e_i >, i = 1, ..., n$.
5. Let $\{e_k\}$ be an orthonormal sequence in a Hilbert space H. For $x \in H$, define $y = \sum_{k=1}^{\infty} < x, e_k > e_k$; show that $(x - y) \perp e_k$ $(k = 1, 2, ...)$.
6. Show that for the sequence space l_2 (a real Hilbert space) its conjugate space l_2* is isomorphic to l_2.