UNIT 6

(***Contents*** : Adjoint of bounded linear operator in a Hilbert space H, Algebra of adjoint operators, proudct of adjoints, self-adjoint operators in H, their algebra, Norm of self-adjoint operator, space of self-adjoint operators, every bounded linear operator in H as a sum of self-adjoint operators, eigen value and eigen vectors of self-adjoint operator.)

§ 6.1 Let H be a complex Hilbert space and let $Bd\alpha(H,H)$ denote the space of all bounded linear operators $T : H \to H$. Take one such $T : H \to H$ as a bounded linear operator. Let $y \in$ H.

Define $f_y : H \to$ scalars by the rule :

$$f_y(x) = < T(x), y > \text{ as } x \in H \qquad \text{........... (1)}$$

Notice that if x_1, $x_2 \in$ H, we have

$$f_y(x_1 + x_2) = < T(x_1 + x_2), y > = < T(x_1) + T(x_2), y > \text{ because } T \text{ is linear}$$

$$= < T(x_1), y > + < T(x_2), y > \text{ using property inner product}$$

$$= f_y(x_1) + f_y(x_2);$$

Similarly $f_y(\alpha x_1) = \alpha f_y(x_1)$ for any scalar α.

That means, f_y is a linear functional over H.

Plus $| f_y(x) | = |< T(x), y >| \le \| T(x) \| \| y \|$ by C-S inequality,

$$\le \| T \| \| x \| \| y \| = (\| T \| \| y \|) \| x \| \text{ for all } x \in H.$$

Therefore, f_y is a bounded linear functional over H, and as we had seen earlier, Riesz representation Theorem says, there is a unique member, say $y^* \in H$ such that

$$f_y(x) = < x, y^* > \qquad \text{........... (2)}$$

where we remember that y^* is determined by f_y. From the text as put up above one sees that given $y \in H$, there is a unique member $y^* \in H$ (via f_y).

Let us define $T^* : H \to H$ by formula :

$$T^*(y) = y^* \text{ as described above} \qquad \text{........... (3)}$$

This operator T^* is called adjoint operator to T in H and as explained above they are connected by relation

$< T(x), y > = < x, T^*(y) >$ from (1), (2) and (3) above **for all $x, y \in H$.**

Explanation : T^* is well defined over H. Because, suppose that for all $x, y \in H$, we have simultaneously

$$< T(x), y > = < x, T^*(y) >$$

and $\quad < T(x), y > = < x, T_1^*(y) >$ for another $T_1 : H \to H$.

Therefore we see $< x, T^*(y) > = < x, T_1^*(y) >$ for all $x, y \in H$.

meaning thereby $T^*(y) = T_1^*(y)$ for $y \in H$. *i.e.* $T^* \equiv T_1^*$

Theorem 6.1.1. $T^* : H \to H$ is a bounded linear operator $\left(T^* \in Bd\alpha(H, H)\right)$.

Proof : Let $x, y, z \in H$. Then $< x, T^*(y+z) > = < T(x), y+z >$

$$= < T(x), y > + < T(x), z > = < x, T^*(y) > + < x, T^*(z) >$$

$$= < x, T^*(y) + T^*(z) >.$$

Therefore, $T^*(y+z) = T^*(y) + T^*(z)$ .... (1)

Again for a scalar λ, $< x, T^*(\lambda y) > = < T(x), \lambda y >$

$$= \bar{\lambda} < T(x), y > = \bar{\lambda} < x, T^*(y) > = < x, \lambda T^*(y) >.$$

Therefore, $T^*(\lambda y) = \lambda T^*(y)$ .... (2)

(1) and (2) together say that T^* is a linear operator.

Again, for $y \in H$ we have

$$\| T^*(y) \|^2 = < T^*(y), T^*(y) > = < T\,T^*(y), y >$$

$$\leq \| T\,T^*(y) \| \| y \| \leq \| T \| \| T^*(y) \| \| y \|$$

That means, $\| T^*(y) \| \leq \| T \| \| y \|$, and therefore T^* is a bound linear operator over H with $\| T^* \| \leq \| T \|$.

Corollary 1. $T^{**} \equiv T$.

Now T^* is a bounded linear operator; and from the relation $< T(x), y > = < x, T^*(y) >$ let us put T^* in place of T to get for all $x, y \in H$,

$$< T^*(x), y > = < x, T^{**}(y) >$$

Interchange x and y to get

$$< T^*(y), x > = < y, T^{**}(x) >$$

Taking conjugates, $< T^{**}(x), y > = < x, T^*(y) > = < T(x), y >$ ....(*)

Now (*) remains true for all $y \in H$, therefore we deduce that

$TT^*(x) = T(x)$ and this being true for all $x \in H$ we finally obtain $T^{**} = T$.

Corollary 2. $\| T^* \| = \| T \|$.

We do already thave $\| T^* \| \le \| T \|$; let us apply this in favour of T^* to get

$$\| T^{**} \| \le \| T^* \|$$

or, $\| T \| \le \| T^* \|$

Therefore, $\| T \| = \| T^* \|$.

§ 6.2 ALGEBRA OF ADJOINT OPERATORS IN HILBERT SPACE *H*.

Let A and B be two bounded linear operators $: H \to H$ *i.e.* $A, B \in Bd\alpha(H, H)$. Then $A + B$ and αA (α any scalar) are also members of $Bd\alpha(H, H)$.

Theorem 6.2.1. (a) $(A+B)^* = A^* + B^*$ and (b) $(\alpha A)^* = \bar{\alpha} A^*$, where A^* denotes adjoint of A.

Proof : (a) For all x, $y \in H$ we have $< A(x), y > = < x, A^*(y) >$ and $< B(x), y > = < x, B^*(y) >$.

Now $< x, (A+B)^* y > = < (a+B)(x), y >$

$$= < A(x) + B(x), y >$$

$$= < A(x), y > + < B(x), y >$$

$$= < x, A^*(y) > + < x, B^*(y) >$$

$$= < x, A^*(y) + B^*(y) >$$

$$= < x, (A^* + B^*)(y) >$$

This shows that $(A+B)^* = A^* + B^*$

(b) $< x, (\alpha A)^*(y) > = < (\alpha A)(x), y >$

$$= < \alpha A(x), y >$$
$$= \alpha < A(x), y >$$
$$= \alpha < x, A^*(y) >$$
$$= < x, \bar{\alpha} A^*(y) >$$
$$= < x, (\bar{\alpha} A^*)(y) >$$

This being true for all $x, y \in H$, we have $(\alpha A)^* = \bar{\alpha} A^*$.

For A and B belonging to $Bd\alpha(H,H)$, let us define $(AB) : H \to H$ by following rule of composition;

$(AB)(x) = A(B(x))$ for $x \in H$. In this way $(BA) : H \to H$ is also defined. It is a routine verification that $(AB) : H \to H$ is a linear operator such that for $x \in H$,

$$\|(AB)(x)\| = \|A(B(x))\| \le \|A\| \|B(x)\| \le \|A\| \|B\| \|x\|.$$

This is true for all $x \in H$; Therefore (AB) is also a bounded linear operator over H *i.e.* $(AB) \in Bd\alpha(H,H)$.

Theorem 6.2.2. $(AB)^* = B^* A^*$.

Proof : For $x, y \in H$, we have $< A(x), y > = < x, A^*(y) >$ and $< B(x), y > = < x, B^*(y) >$

Now $< (AB)(x), y > = < x, (AB)^*(y) >$ which is the same as,

$$< x, (AB)^*(y) > = < (AB)(x), y >$$
$$= < A(B(x)), y >$$
$$= < B(x), A^* y >$$
$$= < x, B^*(A^*(y)) >$$

$= < x, (B^* A^*)(y) >$; Therefore we have $(AB)^* = B^* A^*$.

Theorem 6.2.3. For any $A \in Bd\alpha(H,H)$, $\|A^* A\| = \|A\|^2 = \|AA^*\|$.

Proof : We always have $\|A^* A\| \le \|A^*\| \|A\| = \|A\| \|A\| = \|A\|^2$ because A^* is also a member of $Bd\alpha(H,H)$

i.e. $\|A^*A\| \le \|A\|^2$ .... (1)

Again $\|A\|^2 = \sup_{\|x\|\le 1} \{\|A(x)\|^2\}$

$= \sup_{\|x\|\le 1} \{|<A(x), A(x)>|\}$

$= \sup_{\|x\|\le 1} \{|<A^*(A(x)), x>|\}$

$= \sup_{\|x\|\le 1} \{|<(A^*A)(x), x>|\}$

$\le \sup_{\|x\|\le 1} \{\|(A^*A)(x)\| \, \|x\|\}$ form *C-S* inequality,

$\le \|A^*A\|$.

That is, $\|A\|^2 \le \|A^*A\|$. ....(2)

From (1) and (2) we have $\|A^*A\| = \|A\|^2$. Now applying this equality to A^* one obtains $\|AA^*\| = \|(A^*)^*A^*\| = \|A^*\|^2 = \|A\|^2$. The proof is now complete.

Corollary : If $A \in Bd\alpha(H,H)$ is such that $AA^* = A^*A$ (*i.e.* A and A^* commute), then $\|A^2\| = \|A\|^2$.

§ 6.3 SELF-ADJOINT OPERATORS OVER HILBERT SPACE *H*.

Definition 6.3.1. A member $T \in Bd\alpha(H,H)$ *i.e.* T being a bounded linear operator over H is called self-adjoint if $T^* = T$.

Theorem 6.3.1. (a) If T_1 and T_2 are self-adjoint operators over H, then $T_1 + T_2$ is so.

(b) If T_1 is self-adjoint and α any real scalar, then αT_1 is self-adjoint.

(c) For any member $T \in Bd\alpha(H,H)$, T^*T, TT^* and $T+T^*$ are self-adjoint.

(d) If T_1 and T_2 are self-adjoint, then T_1T_2 is self-adjoint if and only if $T_1T_2 = T_2T_1$ (T_1 and T_2 commute).

Proof : (a) $(T_1 + T_2)^* = T_1^* + T_2^* = T_1 + T_2$

(b) $(\alpha T_1)^* = \bar{\alpha} T_1^* = \bar{\alpha} T_1 = \alpha T_1$ because α is a real scalar.

(c) $(T^*T)^* = T^*\, T T^{**} = T^*T$, $(TT^*)^* = T^{**}T^* = TT^*$;
and $(T + T^*)^* = T^* + T^{**} = T^* + T = T + T^*$.
and finally (d) $(T_1T_2)^* = T_2{}^* T_1{}^* = T_2T_1$; Therefore $(T_1T_2)^* = T_1T_2$ if and only if $T_1T_2 = T_2T_1$.

Theorem 6.3.2. The class of all self-adjoint operators forms a closed real sub-space of $Bd\alpha(H,H)$, and hence it is a Banach space.

Proof : If 0 and I denote the zero operator and identity operator, we have 0 and I are members of $Bd\alpha(H,H)$. Further $0^* = 0$ and $I^* = I$; Now if A and B are self-adjoint operators with α and β two real scalars, we have

$$(\alpha A+\beta B)^* = \bar{\alpha}A^* + \bar{\beta}B^* = \alpha A^* + \beta B^*$$
$$= \alpha A + \beta B$$

Showing thereby that $\alpha A + \beta B$ is also self-adjoint.

Further if $\{A_n\}$ is a sequence of self-adjoint operators over H such that $\lim\limits_{n\to\infty} A_n = A$ in operator norm, *i.e.* $\| A_n - A \| \to 0$ as $n \to \infty$. Then we know that A is a bounded linear operator on Hilbert space. So that

$$\| A - A^* \| \le \| A - A_n \| + \| A_n - A_n{}^* \| + \| A_n{}^* - A^* \|$$
$$= \| A - A_n \| + \| (A_n - A)^* \|$$
$$= \| A - A_n \| + \| A_n - A \| \to 0 \text{ as } n \to \infty.$$

Hence $A = A^*$ and A is self-adjoint.

Theorem 6.3.3. Let A be a bounded linear operator : $H \to H$ such that for all $x, y \in H$, $< A(x), y > = 0$, then A equals to the zero operator and conversely.

Proof : For the zero operator we always have $< 0(x), y > = < \underset{\sim}{0}, y > = 0$. Conversely let for all x, $y \in H$, $<A(x), y> = 0$. Let us fix $x \in H$ and consider $< A(x), y > = 0$ for all $y \in H$. That means $A(x) = \underset{\sim}{0}$ in H; Now let x be free and we see $A(x) = \underset{\sim}{0}$ for $x \in H$; showing $A = 0$.

Corollary : If A is a bounded linear operator : $H \to H$ satisfies $< A(x), x > = 0$ for all $x \in H$, then A is the zero operator.

If $x, y \in H$ and α, β are any two scalars we have

$$0 = <(\alpha x + \beta y), \quad \alpha x + \beta y>$$

$$= <\alpha A(x) + \beta A(y), \quad \alpha x + \beta y> \quad (A \text{ is Linear})$$

$$= \alpha\overline{\alpha} < A(x), x > + \alpha\overline{\beta} < A(x), y > + \beta\overline{\alpha} < A(y), x > + \beta\overline{\beta} < A(y), y >$$

$= \alpha\overline{\beta} < A(x), y > + \beta\overline{\alpha} < A(y), x >$ other terms are zero by given condition.

Let us take $\alpha = 1$ and $\beta = 1$, then we have

$$< A(x), y > \div < A(y), x > = 0 \qquad \text{...........(1)}$$

Again take $\alpha = i$ and $\beta = 1$, then above gives

$$i < A(x), y > - i < A(y), x > = 0$$

or, $$< A(x), y > - < A(y), x > = 0 \qquad (2)$$

Adding (1) and (2) we deduce $< A(x), y > = 0$, and now apply Theorem 6.3.3 for desired conclusion.

Theorem 6.3.4. Let $T \in Bd\alpha(H, H)$ ($T : H \to H$ is a bounded linear operator). Then T is self-adjoint if and only if $< A(x), x >$ is a real scalar for all $x \in H$ (Hilbert space).

Proof : Suppose T is a self-adjoint operator over H, and let $x \in H$; we have

$$\overline{< T(x), x >} = < x, T(x) > = < x, T^*(x) > = < T(x), x >$$

Therefore scalar $< T(x), x >$ is a real scalar.

Conversely, let $< T(x), x >$ is real for all $x \in H$.

Then $< T(x), x > = <\overline{T(x), x}> = <\overline{x, T^*(x)}> = < T^*(x), x >$

Thus $< T(x), x > - < T^*(x), x > = 0$

or, $< T(x) - T^*(x), x > = 0$

or, $< (T - T^*)(x), x > = 0$

This being true for all x in H, we conclude that

$T - T^*$ = zero operator

or, $T = T^*$

i.e. T is a self-adjoint operator.

Theorem 6.3.5. If H is a Hilbert space and $T \in Bd\alpha(H, H)$, such that T is self-adjoint, Then $\| T \| = \sup_{\|x\|=1} |< T(x), x >|$

Proof : If T is self-adjoint, it is ofcourse a bounded linear operator over H. Then for any x with $\| x \| = 1$ in H.

$$|< T(x), x >| \leq \| T(x) \| \| x \| \quad \text{by C-S inequality,}$$

$$\leq \| T \| \| x \| . \| x \| = \| T \| .$$

Therefore, $\sup_{\|x\|\leq 1} |< T(x), x >| \leq \| T \|$ ....(1)

Let K= $\sup_{\|x\|\leq 1} |< T(x), x >|$.

Now we show that $\| T \| \leq K$

If $T(u) = 0$ for all u with $\| u \| = 1$ in H, then we see that $T = 0$ (zero operator), and in that case we have finished.

Otherwise for any z with $\| z \| = 1$ such that $T(z) \neq \underset{\sim}{0}$, put $v = \sqrt{\| T(z) \|}.z$ and $w = \frac{1}{\sqrt{\| T(z) \|}} T(z)$. Then $\| v \|^2 = \| w \|^2 = \| T(z) \|$. Let us now put $y_1 = v + w$ and $y_2 = v - w$. Then on straight calculation and using the fact that T is self-adjoint, we have

$$< T(y_1), y_1 > - < T(y_2), y_2 > = 2(< T(v), w > + < T(w), v >$$

$$= 2(< T(z), T(z) > + < T^2(z), z >) = 4 \| T(z) \|^2 \qquad \text{...........(2)}$$

Now for every $y \neq \underset{\sim}{0}$, and $x = \frac{y}{\| y \|}$, we have

$$y = \| y \| x \text{ and } < T(y), y > = \| y \|^2 |< T(x), x >|$$

$$\leq \| y \|^2 \leq \sup_{\|u\|=1} |< T(u), u >| = K \| y \|^2 .$$

Now $|< T(y_1), y_1 > - < T(y_2), y_2 >| \leq |< T(y_1), y_1 >| + |< T(y_2), y_2 >|$

$$\leq K(\| y_1 \|^2 + \| y_2 \|^2)$$

$$= 2K(\| v \|^2 + \| w \|^2)$$

$$= 4K \| T(z) \|$$

From here and (2) we get $4 \| T(z) \|^2 \leq 4K \| T(z) \|$

Hence $\| T(z) \| \leq K$

So taking supremum over all z with norm | one obtains $\| T \| \leq K$

together with $K \leq \| T \|$ from (1) we finally get $\| T \| = K$.

Theorem 6.3.6. Let $T \in Bd\alpha(H, H)$, H being Hilbert space show that following statements are equivalent.

(a) $T^*T = I$ (Identity operator)

(b) $< T(x), T(y) > = < x, y >$ for all $x, y \in H$

(c) $\| T(x) \| = \| x \|$ for all $x \in H$

Proof : (a) $\Rightarrow$ (b). Let (a) hold. Then for all $x, y \in H$, we have

$$< T^*T(x), y > = < I(x), y > = < x, y >$$

or, $< T(x), T(y) > = < x, y >$, (b) follows.

(b) $\Rightarrow$ (c) ; suppose (b) is true. Taking $y = x$ in (b).

We have $< T(x), T(x) > = < x, x >$

or, $\| T(x) \|^2 = \| x \|^2$

or, $\| T(x) \| = \| x \|$

$(c) \Rightarrow (a)$; Then $\| T(x) \| = \| x \|$ gives $\| T(x) \|^2 = \| x \|^2$

or, $< T(x), T(x) > = < x, x >$

or, $< T^*(T(x)), x > = < x, x >$

or, $< T^*T(x), x > - < x, x > = 0$

or, $\left((T^*T - I)(x), x\right) = 0$; Here we apply corollary of Theorem 6.3.3 to conclude that $T^*T - T = 0$ or, $T^*T = I$.

§ 6.4 EIGEN VALUES AND EIGEN VECTORS OF OPERATOR ON HILBERT SPACE *H*.

Let T be a bounded Linear operator : $H \to H$ *i.e.* $T \in Bd\alpha(H,H)$.

Definition 6.4.1. A non-zero vector $x \in H$ is said to be an eigen vector corresponding to a scalar λ called an eigen value of T if

$$T(x) = \lambda x$$

or, $T(x) - \lambda I(x) = 0$ (I denoting Identity operator on H)

or, $(T - \lambda I)(x) = 0$

Theorem 6.4.1. Let $T : H \to H$ be a self-adjoint operator. Then (1) all eigen values of T (if they exist) are real, and (2) Eigen vectors corresponding to different eigen values of T are orthognonal.

Proof : (a) Let λ be an eigen value of T and x a corresponding eigen vector. Then $x \neq \underset{\sim}{0}$ and $T(x) = \lambda x$.

Since T is self-adjoint, we have

$$\lambda < x, x > = < \lambda x, x > = < T(x), x > = < x, T(x) > = < x, \lambda x >$$

$= \bar{\lambda} < x, x >$ where $< x, x > = \| x \|^2$ is +ve as $x \neq \underset{\sim}{0}$, and this gives $\lambda = \bar{\lambda}$ (since $\| x \| > 0$) and therefore λ is real.

(b) Let λ and μ be two different eigen values of T, and let x and y be eigen vectors (non-zero) corresponding to eigen values λ and μ respectively.

Then we have $T(x) = \lambda x$ and $T(y) = \mu y$. Since T is self-adjoint and eigen values are real, we have

$$\lambda < x, y > = < \lambda x, y > = < T(x), y > = < x, T(y) >$$

$$= < x, \mu y > = \mu < x, y >, \quad \mu \text{ being real.}$$

Since $\lambda \neq \mu$ we conclude that $< x, y > = 0$ or, $x \perp y$ holds.

Theorem 6.4.2. If $T \in Bd\alpha(H,H)$ such that $T^*T = TT^*$, then if x is an eigen vector of T with eigen value λ, then x is also an eigen value of T^* with eigen value $\bar{\lambda}$, and conversely.

Proof : Consider the operator $T - \lambda I$ in H. Then

$$(T - \lambda I)(T - \lambda I)^* = (T - \lambda I)(T^* - \bar{\lambda} I) = TT^* - \bar{\lambda} I - \lambda T^* + | x |^2,$$

and similarly $(T - \lambda I) + (T - \lambda I) = T^*T - \lambda T^* - \bar{\lambda} T + | \lambda |^2 I$

Given $T^*T = T^*T$. Therefore

$$(T-\lambda I)(T-\lambda I)^* = (T-\lambda I)^*(T-\lambda I) \text{ puting } T-\lambda I = S$$

We have $SS^* = S^*S$,

Thus for $x \in H$, $SS^*(x) = S^*S(x)$

or, $\quad < SS^*(x), x > = < S^*S(x), x >$

or, $\quad < S^*(x), S^*x > = < S(x), S(x) >$

or, $\quad \| S^*(x) \|^2 = \| S(x) \|^2$

or, $\quad \| (T^* - \bar{\lambda} I)(x) \|^2 = \| (T-\lambda I)(x) \|^2$

or, $\quad \| (T-\lambda I)(x) \|^2 = \| (T^* - \bar{\lambda} I)(x) \|^2$

or, $\quad \| T - \lambda x \| = \| T^* - \bar{\lambda} x \|$.

This shows that $T(x) = \lambda x$ if and only if $T^*(x) = \bar{\lambda} x$.

Example 6.4.1. Let $L_2[0,1]$ be the real Hilbert space of all square integrable functions over the closed interval [0,1] with I.P. function $< x, y > = \int_0^1 x(t)\, y(t) dt$ as $x, y \in L_2[0,1]$.

Show that $T : L_2[0,1] \to L_2[0,1]$ defined by $T(x) = y \in L_2[0,1]$ where $y(t) = t\, x(t)$ in $0 \le t \le 1$ is a bounded linear operator which is self-adjoint having no eigen values.

Solution : Here T is a linear operator because if $x, y \in L_2[0,1]$ and if $T(x+y) = z$ where $z(t) = t(x+y)(t)$, in $0 \le t \le 1$, we have

$$T(x+y)(t) = z(t) = t\left(x(t) + y(t)\right) = tx(t) + ty(t)$$

$$= T(x)(t) + T(y)(t) \qquad \text{in } 0 \le t \le 1.$$

$\therefore$ $T(x+y) = T(x) + T(y)$ and similarly for any real scalar α, $T(\alpha x) = \alpha\, T(x)$.

Further, $T(x)(t) = tx(t)$ in $0 \le t \le 1$.

$\therefore \; \|T(x)\|^2 = \int_0^1 t^2 x^2(t)dt \le \sup_{0\le t\le}\{t^2\}\int_0^1 x^2(t)dt$

$= 1.\|x\|^2;$

Thus $\|T(x)\| \le \|x\|$; that shows that T is a bounded linear operator in $L_2[0,1]$.

***T* is self-adjoint.** Let $x, y \in L_2[0,1]$, then we have

$$< x, T(y) > = \int_0^1 x(t)ty(t)dt = \int_0^1 tx(t)y(t)dt$$

and $< y, T(x) > = \int_0^1 y(t)t\, x(t)dt = \int_0^1 tx(t)y(t)dt$

Therefore $< x, T(y) > = < y, T(x) >$; That shows T as self-adjoint.

If λ is an eigen value of T, and a non-zero $x \in L_2[0,1]$ is an eigen vector of T corresponding to the eigen value λ, we have

$T(x) = \lambda x$

or, $tx(t) = \lambda x(t)$ in $0 \le t \le 1$

or, $(t - \lambda)x(t) = 0$ in $0 \le t \le 1$

Since x is non-zero, we have $t = \lambda$ in $0 \le t \le 1$, which is not the case. Thus no such λ is there, *i.e.* T possesses no eigen value.

Theorem 6.4.3. Every bounded linear operator T on a Hilbert space H is equal to a sum $A + iB$ where A and B are self-adjoint operator in H.

Proof : Let us define A and B as follows :

$$A = \frac{1}{2}(T + T^*), \text{ and } B = \frac{1}{2i}(T - T^*).$$

Then $A^* = \frac{1}{2}(T^* + T) = A$ and $B^* = -\frac{1}{2i}(T^* - T) = \frac{1}{2i}(T - T^*) = B$; So each of A and B is a self-adjoint operator on H such that $A + iB = T$.

Remark : Representation of T as $T = A + iB$ is unique. Because, Let $T = C + iD$ where C and D are self-adjoint operator on H; then $T^* = (C + iD)^* = C - iD$ and hence $T + T^* = 2C$ and $T - T^* = 2iD$; Thus $C = A$ and $D = B$.

EXERCISE A

Short answer type questions

1. Find the eigen values and eigen vectors of $\begin{pmatrix} a, & b \\ -b, & a \end{pmatrix}$ $b \neq 0$ and *a, b* are reals.
2. Examine if zero operator and Identity operator in a Hilbert space *H* are self-adjoint.
3. If *T* is a self-adjoint operator in a Hilbert space *H*, show that for every natural number *n*, T^n is self-adjoint.
4. If *T* is a self-adjoint operator in a Hilbert space *H*, and *S* is any bounded Linear operator in *H*, show that *S*TS* is self-adjoint.
5. Show that $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ does not possess any eigen vector.

EXERCISE B

1. Given a square matrix $A = \left((a_{ji})\right)_{n \times n}$ having eigen values $\lambda_1, \lambda_2, ..., \lambda_n$, show that *kA* has eigen values $k\lambda_1, k\lambda_2, ..., k\lambda_n$; and A^2 has eigen values $\lambda_1^2, \lambda_2^2, ...\lambda_n^2$.
2. Let $T: l_2 \to l_2$ be defined by $T(\xi_1, \xi_2, ...) = (0, 0, \xi_1, \xi_2, ...)$ as $(\xi_1, \xi_2, ...) \in l_2$; Examine if *T* is a bounded linear operator in l_2 and if *T* is self-adjoint in l_2.
3. Show that in a Hilbert space *H*, $T_1{}^* T_1 = T_2{}^* T_2$ if and only if $\| T_1(x) \| = \| T_2(x) \|$ for all $x \in H$.
4. In *H* if *T* is self-adjoint show that $T(x) = \underset{\sim}{0}$ in *H* if and only if $T\,T(x) = \underset{\sim}{0}$.
5. Let $T : H \to H$ and $W : H \to H$ be bounded Linear operators and $S = W^*TW$. Show that if *T* is self-adjoint and +ve, so will be *S*.

NOTES