Unit 1 □ Irrotational Motion of an Ideal Fluid in Two-Dimensions

Structure

1.0 Introduction

In this chapter, we consider the two-dimensional irrotational steady flow of an ideal incompressible fluid. For plane flow, all dynamic computations for the hydrodynamic considerations, we take a layer of unit height cut by two planes parallel to the plane of the flow. In considering the plane problem, we direct our attention on the kinetic flow around a body fixed in a flow or for the motion of a body in a fluid at rest. We shall restrict our discussions on cylindrical bodies having circular and elliptic cross-sections.

1.1 Irrotational Motion in Two Dimensions. The Stream Function

If the motion of a liquid remains the same in all planes parallel to that of xy and there is no velocity parallel to the z-axis, i.e. if the velocity components u, v are functions of x, y only and the component w = 0, then the motion is said to be two-dimensional and in such a case, we consider the circumstances in the xy-plane. When we speak of the flow across a curve in this plane, we mean the flow is across a unit length of a cylinder whose trace on the xy plane is the curve in question, the generators of the cylinder being parallel to the axis of z. Here the differential equation of the lines of flow is

$$vdx - udy = 0 \quad (1)$$

while the equation of continuity is

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \text{ i.e. } \frac{\partial u}{\partial x}+\frac{\partial}{\partial y}=(-v). \quad (2)$$

This equation shows that the left hand side of (1) is an exact differential $d\psi$, say. Thus

$$vdx - udy = d\psi = \frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy$$

leading to $$u = -\frac{\partial \psi}{\partial y},\ v = \frac{\partial \psi}{\partial x} \tag{3}$$

This function $\psi(x, y)$ is called the **stream function** or **current function**. It follows that the lines of flow are given by ψ = constant.

Now if the motion of the liquid be irrotational, then there exists a velocity potential $\phi(x, y)$ such that

$$u = -\frac{\partial \phi}{\partial x},\ v = -\frac{\partial \phi}{\partial y} \tag{4}$$

From (3) and (4) we get

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y},\ \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} \tag{5}$$

so that

$$\frac{\partial \phi}{\partial x}\frac{\partial \psi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{\partial \psi}{\partial y} = 0$$

which shows that the families of curves ϕ = constant, ψ = constant cut orthogoralIy at all their points of intersection. These conditions are satisfied if we take $\phi + i\psi$ to be a function of the complex variable $x + iy$.

Now let $\phi + i\psi = f(x + iy)$. Then

$$\frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x} = f'(x+iy),\ \frac{\partial \phi}{\partial y} + i\frac{\partial \psi}{\partial y} = if'(x+iy) = i\frac{\partial \phi}{\partial x} - \frac{\partial \psi}{\partial x}$$

giving $$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y},\ \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$$

Thus ϕ and ψ are conjugate functions. If $w = \phi + i\psi = f(z)$, then w is called the **complex potential**.

Noting that

$$\frac{dw}{dz} = \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x} = \frac{\partial \phi}{\partial x} - i\frac{\partial \phi}{\partial y} = -u + iv \tag{6}$$

we have the magnitude of the velocity at any point as $\left|\frac{dw}{dz}\right|$, since

$$\left|\frac{dw}{dz}\right| = \left\{\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2\right\}^{\frac{1}{2}} = (u^2 + v^2)^{\frac{1}{2}} = \text{velocity}. \tag{7}$$

1.2 Boundary Conditions

From (5), it follows that

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -\frac{\partial^2 \phi}{\partial x \partial y} + \frac{\partial^2 \phi}{\partial y \partial x} = 0$$

where it is assumed the validity of $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$. Thus the stream function ψ must satisfy the Laplace's equation

$$\nabla^2 \psi = 0 \tag{8}$$

at all points of the liquid. This function ψ satisfies the following boundary conditions :

(a) If the liquid is at rest at infinity, we must have $\frac{\partial \psi}{\partial x} = 0$ and $\frac{\partial \psi}{\partial y} = 0$ at infinity.

(b) At any fixed boundary, the normal velocity must be zero, or the boundary must coincide with a stream line ψ = constant.

(c) At the boundary of the moving cylinder, the normal component of the velocity of the liquid must be equal to the normal component of the velocity of the cylinder.

We now express the condition (c) by a formula for ψ as follows.

Let a point O of the cross-section of any cylinder be taken as origin. Let U and V be the velocities parallel to the axis of x and y at O and let the cylinder turn with the angular velocity ω. If P(x, y) be any point on the surface of the cylinder, then the velocity components of P are $U - \omega y$ and $V + \omega x$. If θ is the inclination of the tangent at P with Ox, then from the differential calculus, we have

$$\cos\theta = \frac{dx}{ds} \text{ and } \sin\theta = \frac{dy}{ds} \tag{9}$$

Therefore, the outward normal velocity at P

$$= (U - \omega y) \sin\theta - (V + \omega x) \cos\theta$$

$$= (U - \omega y)\frac{dy}{ds} - (V - \omega x)\frac{dx}{ds}. \tag{10}$$

Also the velocity of the liquid in the outward normal is $-\frac{\partial \psi}{\partial s}$.

On equating above two expressions for the normal component of velocities in accordance with condition (c), we have

$$-\frac{\partial\psi}{\partial s} = (U - \omega y)\frac{dy}{ds} - (V - \omega x)\frac{dx}{ds}.$$

Integrating this equation along the arc, we get

$$\psi = Vx - Uy + \frac{1}{2}\omega(x^2 + y^2) + C \tag{11}$$

where C is an arbitrary constant.

Let the cylinder move along the x-axis with velocity U without rotation (so that V = 0 and ω = 0). Then (11) reduces to

$$\psi = -Uy + C. \tag{12}$$

Similarly, if the cylinder moves along the y-axis with velocity V without rotation, then (11) gives

$$\psi = Vx + C. \tag{13}$$

1.3 Motion of a Circular Cylinder

Let a circular cylinder of radius a is moving in an infinite mass of liquid at rest at infinity, with velocity U in the direction of x-axis. To find the velocity potential ϕ that will satisfy the given boundary conditions, we have the following conditions :

(*i*) ϕ satisfies the Laplace's equation

$$\nabla^2\phi = 0$$

at every point of the liquid. In polar co-ordinates (r, θ) in two dimensions, $\nabla^2\phi = 0$ takes the form

$$\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 0. \tag{14}$$

which has solutions of the form

$$r^n \cos n\theta, \quad r^n \sin n\theta,$$

where n is any integer, positive or negative. Hence the sum of any number of terms of the form

$$A_n r^n \cos n\theta,\ B_n r^n \sin n\theta$$

is also a solution of (14).

(*ii*) Normal velocity at any point of the cylinder = Velocity of the liquid at that point in that direction, i.e.,

$$-\frac{\partial\phi}{\partial r} = U\cos\theta \qquad \text{when } r = a. \tag{15}$$

(*iii*) Since the liquid is at rest at infinity, velocity must be zero there. Thus,

$$-\frac{\partial\phi}{\partial r} = 0 \text{ and } -\frac{1}{r}\frac{\partial\phi}{\partial r} = 0 \text{ at } r = \infty. \tag{16}$$

The above considerations suggest that we must assume the following suitable form of ϕ.

$$\phi = Ar\cos\theta + \frac{B}{r}\cos\theta. \tag{17}$$

From (17)

$$-\frac{\partial\phi}{\partial r} = -\left(A - \frac{B}{r^2}\right)\cos\theta. \tag{18}$$

so that using (15), we get

$$U\cos\theta = -\left(A - \frac{B}{a^2}\right)\cos\theta, \text{ valid for all values of } \theta.$$

Hence,

$$-U = \left(A - \frac{B}{a^2}\right).$$

Again the first condition of (16) gives $A = 0$. Thus $B = Ua^2$.
Hence (17) reduces to

$$\phi = \frac{Ua^2}{r}\cos\theta. \tag{19}$$

It may be noted that (19) also satisfies the second condition given by (16). Hence (19) gives the required velocity potential. But

$$\frac{\partial\psi}{\partial r} = -\frac{1}{r}\frac{\partial\phi}{\partial\theta} = \frac{Ua^2}{r^2}\sin\theta$$

After integrating, we obtain

$$\psi = -\frac{Ua^2}{r}\sin\theta \tag{20}$$

which gives the stream function of the motion. The complex potential w is given by

$$w = \frac{Ua^2}{r}(\cos\theta - i\sin\theta) = \frac{Ua^2}{z} \tag{21}$$

where $z = re^{i\theta}$.

1.4 Fixed Circular Cylinder in a Uniform Stream

Let a circular cylinder be fixed at the origin and x-axis be chosen in the opposite direction of the stream U. Let R′ be the region $r \geq a$. Now the velocity potential ϕ satisfies the relation

$$\nabla^2 \phi = 0 \text{ in } R'. \tag{22}$$

The boundary conditions are

$$\phi \sim Ux \text{ at infinity,}$$

and

$$-\frac{\partial\phi}{\partial r} = 0 \text{ on the boundary of cylinder.}$$

Let us take

$$\phi = Ur\cos\theta + \phi_1, \tag{23}$$

where ϕ_1 is the contribution due to pressure of the cylinder.

The boundary conditions give

$$\phi_1 \to 0 \text{ at infinity} \tag{24}$$

and

$$-\frac{\partial\phi_1}{\partial r} = U\cos\theta \text{ on } C : r = a. \tag{25}$$

Now, since ϕ is harmonic, so ϕ_1 is harmonic and its normal derivative is prescribed on the boundary.

Now let us assume ϕ_1 to be of the form

$$\phi_1 = \left(Ar + \frac{B}{r}\right)\cos\theta.$$

To satisfy the condition (24), we have A = 0 and from (95), we get B = a^2U. Hence

$$\phi = Ur\cos\theta + \frac{Ua^2}{r}\cos\theta. \tag{26}$$

Again, we have

$$\frac{\partial\psi}{\partial r} = -\frac{1}{r}\frac{\partial\phi}{\partial\theta},$$

which gives

$$\psi = Ur\sin\theta - \frac{Ua^2}{r}\sin\theta. \tag{27}$$

Hence, the complex potential is

$$w(z) = Uz + \frac{Ua^2}{z} \text{ in } R'. \tag{28}$$

The equation of stream line is

$$\psi = \text{constant}$$

or,

$$\left(r - \frac{a^2}{r}\right)\sin\theta = \text{constant}$$

or,

$$\left(y - \frac{a^2 y}{x^2 + y^2}\right) = \text{constant}. \tag{29}$$

Complex velocity is given by

$$-\frac{dw}{dz} = -V_0\left(1 - \frac{a^2}{z^2}\right) \tag{30}$$

Then $\frac{dw}{dz} = 0$ implies

$$z = \pm a.$$

Therefore $z = \pm a$ are stagnation points (a point where the velocity is zero is called a stagnation point. The stream lines are not well-defined thereat; a stream line may divide into two branches at such a point).

1.5 Circulation About a Circular Cylinder

If A and P be any two points in a liquid, then $\int_A^P (udx + vdy + wdz)$ is called the **flow** along the path from A to P, where u, v, w are velocity components. If the velocity potential ϕ exists, i.e. if the motion be irrotational, then

$$u = -\frac{\partial\phi}{\partial x},\ v = -\frac{\partial\phi}{\partial y},\ w = -\frac{\partial\phi}{\partial z}$$

and so

$$\text{flow} = -\int_A^P \left(\frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz \right) = \phi_A - \phi_P$$

The flow round a closed curve C is known as **circulation** which is usually denoted by Γ. Thus

$$\Gamma = -\oint_C (udx + vdy + wdz).$$

If the motion is irrotational and the velocity potential ϕ is single-valued, then circulation round C is zero.

Let k be the constant circulation about the cylinder. Then the suitable form of ϕ in two dimensions (r, θ) may be obtained by equating to k the circulation round a circle of radius r. Thus, we have

$$\left(-\frac{\partial\phi}{r\partial\theta} \right)(2\pi r) = k,$$

integrating this we get

$$\phi = -\frac{k\theta}{2\pi}.$$

Again,

$$\frac{\partial\psi}{\partial r} = \frac{\partial\phi}{r\partial\theta},$$

which gives

$$\psi = \frac{k}{2\pi}\ln r,$$

Thus the complex potential due to the circulation about a circular cylinder is given by

$$w = \frac{ik}{2\pi}(\ln r + i\theta)$$

or,

$$w = \frac{ik}{2\pi}\ln z, \text{ (since } z = re^{i\theta}.) \tag{31}$$

1.6 Steaming and Circulation About a Fixed Circular Cylinder

We know that the complex potential w_1 due to the circulation of strength k about the cylinder is given by

$$w_1 = \frac{ik}{2\pi}\ln z.$$

Also, the complex potential w_2 for streaming past a fixed circular cylinder of radius a, with velocity U in the negative direction of x-axis is given by

$$w_2 = Uz + \frac{Ua^2}{z}.$$

Thus, the complex potential w due to the combined effects at any point z is given by

$$w = w_1 + w_2$$

$$= U\left(z + \frac{a^2}{z}\right) + \frac{ik}{2\pi}\ln z. \tag{32}$$

$$\text{i.e. } \phi + i\psi = U\left(re^{i\theta} + \frac{a^2}{r}e^{-i\theta}\right) + \frac{ik}{2\pi}\ln(re^{i\theta})$$

Equating real and imaginary parts, we obtain

$$\phi = U\left(r + \frac{a^2}{r}\right)\cos\theta - \frac{k\theta}{2\pi} \tag{33}$$

and

$$\psi = U\left(r - \frac{a^2}{r}\right)\sin\theta + \frac{k}{2\pi}\ln r.$$

Since the velocity will be tangential only at the boundary of the cylinder, so $-\left(\frac{\partial\phi}{\partial r}\right) = 0$ and hence the magnitude of the velocity $\bar{q}$ is given by

$$\left| -\frac{\partial\phi}{r\partial\theta} \right| = \left| 2\,U\sin\theta + \frac{k}{2\,\pi a} \right|.$$

If there is no circulation, i.e. if k = 0 there would be points of zero velocity on the cylinder at $\theta = 0$ and $\theta = \pi$, the former being the point at which the incoming stream divides. However, in the presence of circulation, the stagnation points are given by $q = 0$, i.e.

$$\sin\theta = -\frac{k}{4\,\pi Ua}$$

and such points exist when

$$|\,k\,| < 4\,\pi Ua. \tag{34}$$

We now determine the pressure at points of the cylinder. The pressure is given by Bernoulli's equation

$$\frac{p}{\rho} = C(\,t\,) - \frac{1}{2}q^2. \tag{35}$$

Let Π be the pressure at infinity where the velocity is U and so

$$\frac{\Pi}{\rho} = C(\,t\,) - \frac{1}{2}U^2.$$

Then from (35) we obtain

$$\frac{p}{\rho} = \frac{\Pi}{\rho} + \frac{1}{2}(\,U^2 - q^2\,)$$

or,

$$p = \Pi + \frac{1}{2}\rho U^2 - \frac{1}{2}\rho\left(\,2\,U\sin\theta + \frac{k}{2\,\pi a}\,\right)^2. \tag{36}$$

If X, Y be the components of the thrust on the cylinder, we have

$$X = -\int_0^{2\pi} p\cos\theta a d\theta,$$

$$Y = -\int_0^{2\pi} p\sin\theta a d\theta.$$

Using (36) we get $X = 0$, $Y = \rho kU$, showing that the cylinder experiences an upward lift. This effect may be attributed to circulation phenomenon.

1.7 Equation of Motion of a Circular Cylinder

Let a circular cylinder is moving in a liquid at rest at infinity. To calculate the forces acting on the cylinder owing to the pressure of the fluid, we suppose that U, V are the components of the velocity of the cylinder when the center of the cross-section O is (x_0, y_0). Then we have

$$U = \dot{x}_0 \text{ and } V = \dot{y}_0.$$

Let $z_0 = x_0 + iy_0$ and $z - z_0 = re^{i\theta}$ where r denotes the distance from the axis of the cylinder.

On the surface of the cylinder r = a, we must have, the velocity of the liquid normal to the cylinder = normal velocity of the cylinder, i.e.

$$-\frac{\partial \phi}{\partial r} = U\cos\theta + V\sin\theta \text{ at } r = a. \tag{37}$$

Since the liquid is at rest at infinity,

$$-\frac{\partial \phi}{\partial r} = 0 \text{ as } r \to \infty. \tag{38}$$

The conditions (37) and (38) suggest that ϕ is to be taken in the form

$$\phi = \left(Ar + \frac{B}{r}\right)\cos\theta + \left(Cr + \frac{D}{r}\right)\sin\theta. \tag{39}$$

Therefore

$$\frac{\partial \phi}{\partial r} = \left(A - \frac{B}{r^2}\right)\cos\theta + \left(C - \frac{D}{r^2}\right)\sin\theta.$$

Using (37) and (38) we get

$$U = \frac{B}{a^2} - A,\ V = \frac{D}{a^2} - C,\ A = C = 0$$

Thus we have $B = a^2U$, $D = a^2V$,

Hence from (39), the expression for ϕ is given by

$$\phi = \frac{a^2}{r}(U\cos\theta + V\sin\theta). \tag{40}$$

Noting that

$$\frac{\partial \psi}{\partial r} = -\frac{\partial \phi}{r\partial \theta},$$

and using (40) and then integrating this equation, we obtain

$$\psi = \frac{a^2}{r}(-U\sin\theta + V\cos\theta). \tag{41}$$

Hence the complex potential is given by

$$w = \phi + i\psi = \frac{a^2 e^{-i\theta}}{r}(U + iV),$$

i.e.

$$w = \frac{a^2(U + iV)}{z - z_0}. \tag{42}$$

Now

$$\frac{\partial\phi}{\partial t} + i\frac{\partial\psi}{\partial t} = \frac{\partial w}{\partial t} = \frac{a^2(\dot{U} + i\dot{V})}{z - z_0} + \frac{a^2(U + iV)^2}{(z - z_0)^2}. \tag{43}$$

Equating real parts, we obtain

$$\frac{\partial\phi}{\partial t} = \frac{a^2}{r}(\dot{U}\cos\theta + \dot{V}\sin\theta) + \frac{a^2}{r^2}\left[(U^2 - V^2)\cos 2\theta + 2UV\sin 2\theta\right]. \tag{44}$$

The magnitude of the velocity $\vec{q}$ is given by

$$q^2 = \left|\frac{dw}{dz}\right|^2 = \left|-a^2\frac{U + iV}{(z - z_0)^2}\right|^2 = \frac{a^4(U^2 + V^2)}{r^4}. \tag{45}$$

Omitting the external forces, the pressure at any point is given by Bernoulli's equation as

$$\frac{p}{\rho} = C(t) + \frac{\partial\phi}{\partial t} - \frac{1}{2}q^2.$$

which, on using (44) and (45) gives

$$\frac{p}{\rho} = C(t) + \frac{a^2}{r}\left(\dot{U}\cos\theta + \dot{V}\sin\theta\right) + \frac{a^2}{r^2}\left[(U^2 - V^2)\cos 2\theta + 2UV\sin 2\theta\right] - \frac{1}{2}\frac{a^4}{r^4}(U^2 - V^2) \tag{46}$$

Let p_1 be the pressure at a point (a, θ) on the boundary of the cylinder. Then p_1 is given by (46) on putting $r = a$ as

$$p_1 = \rho C(t) + \rho a\left(\dot{U}\cos\theta + \dot{V}\sin\theta\right) + \rho\left[(U^2 - V^2)\cos 2\theta + 2UV\sin 2\theta\right] - \frac{1}{2}(U^2 + V^2). \tag{47}$$

Let X and Y be the components of force on the cylinder due to fluid thrusts. Then, we have

$$X = -\int_0^{2\pi} ap_1 \cos\theta \, d\theta,$$

$$Y = -\int_0^{2\pi} ap_1 \sin\theta \, d\theta.$$

which, with the help of (47), give

$$X = -\rho a^2 \int_0^{2\pi} \dot{U} \cos^2 \theta \, d\theta$$

$$= -\pi a^2 \rho \dot{U}$$

$$= -M' \dot{U},$$

where $M' = \pi a^2 \rho$ = the mass of the liquid displaced by the cylinder of unit length.

Similarly,

$$Y = -\pi a^2 \rho \dot{V} = -M' \dot{V}.$$

Corollary :

To show that the effect of the pressure of the liquid is to reduce the extraneous forces in the ratio

$$(\sigma - \rho) : (\sigma + \rho)$$

where σ, ρ are the densities of the cylinder and liquid respectively, we proceed as follows :

Let M be the mass of the cylinder per unit length and X′, Y′ be the components of the extraneous force on the cylinder if there were no liquid. Also let f_x be the acceleration of the extraneous force in x-direction. Then, due to presence of liquid, the resultant force in x-direction is

$$= \pi a^2 \sigma f_x - \pi a^2 \rho f_x$$

$$= \frac{\sigma - \rho}{\sigma} (\pi a^2 \sigma f_x)$$

$$= \frac{\sigma - \rho}{\sigma} X'.$$

Thus the equation of motion in x-direction is of the form

$$M\dot{U} = -M'\dot{U} + \frac{\sigma - \rho}{\sigma} X'$$

or,

$$M\dot{U} = \frac{M}{M+M'}\cdot\frac{\sigma-\rho}{\sigma}X'$$

or,

$$M\dot{U} = \frac{\pi a^2\sigma}{\pi a^2\sigma+\pi a^2\rho}\cdot\frac{\sigma-\rho}{\sigma}X'.$$

Therefore

$$M\dot{U} = \frac{\sigma-\rho}{\sigma+\rho}X'.$$

Similarly,

$$M\dot{V} = \frac{\sigma-\rho}{\sigma+\rho}Y'.$$

Hence the effect of the pressure of the liquid is to reduce the external force in the ratio

$$(\sigma - \rho) : (\sigma + \rho).$$

1.8 Two Coaxial Circular Cylinders

We now determine the velocity potential and the stream function at any point of a liquid contained between two coaxial circular cylinders of radii a and b(a < b). Let the cylinders are moved suddenly parallel to themselves in directions at right angles with velocities U and V respectively

Then if ϕ be the velocity potential and ψ the stream function at any point (r, θ) in the liquid, then the boundary conditions for the velocity potential ϕ are :

$$-\frac{\partial\phi}{\partial r} = U\cos\theta, \text{ when } r = a$$

and

$$-\frac{\partial\phi}{\partial r} = V\sin\theta, \text{ when } r = b. \tag{48}$$

Now ϕ must satisfy the Laplace's equation

$$\frac{\partial^2\phi}{\partial^2 r} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial^2\theta} = 0. \tag{49}$$

at every point of the liquid.

Since (49) has solutions of the form $r^n \cos n\theta$, $r^n \sin n\theta$, where n is any positive or negative integer, the sum of any number of terms of the form $A_n r^n \cos n\theta$, $B_n r^n \sin n\theta$ is also solution of (49). However, a suitable form of ϕ satisfying the given conditions is

$$\phi = \left(Ar + \frac{B}{r} \right) \cos\theta + \left(Cr + \frac{D}{r} \right) \sin\theta. \tag{50}$$

Using the two boundary conditions (48) we obtain for any values θ

$$A - \frac{B}{a^2} = U,\; C - \frac{D}{a^2} = 0, \text{ and } A - \frac{B}{a^2} = 0,\; C - \frac{D}{a^2} = -V.$$

These give

$$A = \frac{Ua^2}{(b^2 - a^2)},\; B = \frac{Ua^2 b^2}{(b^2 - a^2)},\; C = -\frac{Vb^2}{(b^2 - a^2)},\; D = -\frac{Va^2 b^2}{(b^2 - a^2)}.$$

Thus

$$\phi = \frac{Ua^2}{(b^2 - a^2)} \left(r + \frac{b^2}{r} \right) \cos\theta - \frac{Vb^2}{(b^2 - a^2)} \left(r + \frac{a^2}{r} \right) \sin\theta. \tag{51}$$

Since

$$\frac{\partial \phi}{\partial r} = \frac{\partial \psi}{r \partial \theta}.$$

we get by using (51)

$$\psi = \frac{Ua^2}{(b^2 - a^2)} \left(r - \frac{b^2}{r} \right) \sin\theta + \frac{Vb^2}{(b^2 - a^2)} \left(r - \frac{a^2}{r} \right) \cos\theta. \tag{52}$$

1.9 The Milne-Thomson's Circle Theorem

Statement : *Let f(z) be the complex velocity potential for the two-dimensional irrotational flow of an incompressible inviscid fluid having no rigid boundaries and such that there are no singularities of flow within the circle* $|z| = a$. *Then, on* *introducing the solid circular cylinder* $|z| = a$ *into the flow, the new complex velocity potential is given by* $w = f(z) + \bar{f}(a^2/z)$ *for* $|z| \geq a$.

Proof : Since the singularities of f(z) occur in the region $|z| > a$, so the singularities of $f(a^2/z)$ lie in $|z| < a$. Hence the singularities of $\bar{f}(a^2/z)$ also lie in $|z| < a$. Thus f(z) and $f(z) + \bar{f}(a^2/z)$ both have the same singularities in the region $|z| > 0$ and, therefore, both functions, considered as complex velocity potentials, may be ascribed to the same hydrodynamical distributions in the region $|z| > a$.

Now, on the circle $|z| = a$, we take $z = ae^{i\theta}$, so that $a^2/z = ae^{-i\theta}$ and, therefore,

$$w = f(z) + \bar{f}(a^2/z) = f(ae^{i\theta}) + \bar{f}(ae^{-i\theta}) = f(ae^{i\theta}) + \overline{f(ae^{i\theta})}$$

Thus, on the circle $|z| = a$, w is the sum of a complex quantity and its complex conjugate and is, therefore, w is a real number, i.e. $\psi = \text{Im}(w) = 0$ on $|z| = a$. Hence, the circular boundary is a stream line across which no fluid flows. We, therefore, conclude that $|z| = a$ is a possible boundary for the new flow for which $w = f(z) + \bar{f}(a^2/z)$ is the appropriate complex velocity potential.

Applications of circle theorem :

Example 1. Uniform flow post a stationary cylinder

We have already seen in Section-1.4 that a uniform stream having velocity –U along the negative direction of x-axis gives rise to a complex potential Uz. Thus, if we take $f(z) = Uz$, then $\bar{f}(a^2/z) = \frac{Ua^2}{z}$. Thus on introducing the circular section $|z| = a$ into the stream, the complex potential for the region $|z| \geq a$ is given by

$$w = f(z) + \bar{f}(a^2/z) = U\left(z + \frac{a^2}{z}\right).$$

If $z = re^{i\theta}$ and $w = \phi + i\psi$, then

$$\phi = U\cos\theta\left(r + \frac{a^2}{r}\right), \quad \psi = U\sin\theta\left(r - \frac{a^2}{r}\right)$$

which are the results obtained in Section—1.4.

Example 2. Uniform stream at incidence with the positive x-axis

The complex potential for such a stream of velocity U is $Uze^{-i\beta}$. Thus, if we take f(z)

$= Uze^{-i\beta}$, then $\bar{f}\left(\frac{a^2}{z}\right) = Ue^{i\beta}.\frac{a^2}{z}$. Hence, when the cylinder of section $|z| = a$ is introduced, the complex potential in $|z| \geq a$ becomes $w = U\left\{ze^{-i\beta} + \left(\frac{a^2}{z}\right)e^{i\beta}\right\}$.

1.10 Theorem of Blasius

Statement : *Suppose that, in a steady two-dimensional irrotational motion given by the relation $w = f(z)$, i.e. $\phi + i\psi = f(\alpha + iy)$, the hydrodynamical pressures on the contour of a fixed cylinder are (X, Y) and a couple N about the origin of coordinates. Then*

$$X - ir = \frac{1}{2} i\rho \oint_C \left(\frac{dw}{dz}\right)^2 dz,$$

and

$$M = Re\left\{-\frac{1}{2}\rho \oint_C z\left(\frac{dw}{dz}\right)^2 dz\right\} \qquad (53)$$

where ρ is the density and the integrations are taken round any contour surrounding the cylinder.

Proof : Let the normal to the cylinder at the point $P(x, y)$ make an angle θ with the positive direction of x-axis.

Then, for the action on the arc ds and P, we have

$$dX = -p \sin\theta \, ds, \; dY = p \cos\theta \, ds$$

$$\text{i.e. } dX = -pdy, \; dY = pdx$$

so that

$$X = \oint_{C'} pdy, \; Y = \oint_{C'} pdx.$$

and, therefore,

$$X - iY = -i \oint_{C'} p(dx - idy).$$

where the integrals are round the contour C′ of the cylinder.

Since there is no external force and the fluid is moving irrotationally and steadily, so the pressure equation is given by

$$\frac{p}{q}+\frac{1}{2}q^2 = \text{constant} = A.$$

Thus

$$X - iY = -i\oint_{C'}\rho\left(A-\frac{1}{2}q^2\right)(dx - idy)$$

$$= \frac{i\rho}{2}\oint_{C'}\left(\frac{dw}{dz}\right)^2 d\bar{z}$$

$$= \frac{i\rho}{2}\oint_{C'}\left(\frac{dw}{dz}\right)\left(\overline{\frac{dw}{dz}}\right)d\bar{z}$$

$$= \frac{i\rho}{2}\oint_{C'}\left(\frac{dw}{dz}\right)^2 d\bar{w}.$$

Now the contour of the cylinder is a stream line, i.e. on C′, ψ = constant. Also

$$dw = d\bar{w}.$$

Therefore

$$X - iY = \frac{i\rho}{2}\oint_{C'}\left(\frac{dw}{dz}\right)^2 dz.$$

Now in the plane outside the cylinder, it may be possible to have singularity in the function $\left(\frac{dw}{dz}\right)^2$ if there is any physical singularity in the fluid (such as a source or a vortex). Thus, if we take a larger countour C surrounding C′ such that there are no singularities between C and C′; or more generally, if such singularities exist, then the sum of the residues of $\left(\frac{dw}{dz}\right)^2$ at all poles between C and C′ is zero, then the integrals of this function have the same value for all such contours and we have

$$X - iY = \frac{1}{2}i\rho\int_C\left(\frac{dw}{dz}\right)^2 dz.$$

Again

$$N = \oint_{C'}(pxdx + pydy)$$

$$= \text{Real part of } \oint_{C'} p\,(x + iy)(dx - idy)$$

$$= \text{Real part of } \oint_{C'} pz d\bar{z}$$

$$= \text{Real part of } \left[-\frac{1}{2}\rho \oint_{C'} z \left(\frac{dw}{dz} \right)^2 d\bar{z} \right]$$

$$= \text{Real part of } \left[-\frac{1}{2}\rho \oint_{C'} z \left(\frac{dw}{dz} \right) d\bar{w} \right]$$

$$= \text{Real part of } \left[-\frac{1}{2}\rho \oint_{C'} z \left(\frac{dw}{dz} \right) dw \right]$$

$$= \text{Real part of } \left[-\frac{1}{2}\rho \oint_{C'} z \left(\frac{dw}{dz} \right)^2 dz \right].$$

Considering the same limitation as before regarding singularities in the liquid, the integral may be taken round any contour C which surrounds the cylinder.

1.11 Transformations or Mapping

The set of equations

$$u = u(x, y),\ v = v(x, y) \tag{54}$$

defines, in general, a transformation or mapping which establishes a correspondence between points in the uv- and xy-planes. The equations (54) are called transformation equations. If to each point of the uv-plane there corresponds one and only one point of the xy plane and conversely, we speak of a one-to-one transformation or mapping.

Conformal mapping

Suppose that under the transformation (54), the point (x_0, y_0) of the xy-plane is mapped into the point (u_0, v_0) of the uv-plane while curves C_1 and C_2 [intersecting at (x_0, y_0)] are mapped respectively into curves C'_1 and C'_2. Then, if the transformation is such that the angle at (x_0, y_0) between C_1 and C_2 is equal to the angle at (u_0, v_0) between C''_1 and C''_2 both in magnitude and sense, the transformation or mapping is said to be *conformal* at (x_0, y_0). A mapping which preserves the magnitudes of angles but not necessarily the sense is called *isogonal.*

1.12 The Schwarz-Christoffel Transformations

Any simple closed polygon with n vertices in the z-plane (z = x + iy) can be transformed into the real axis in the $\zeta\,(=\xi + i\eta)$-plane, the interior points of the polygon corresponding to points on one side of the real axis $\eta = 0$, the transformation-effective relation being

$$\frac{dz}{d\zeta} = A(\zeta - a_1)^{\frac{\alpha_1}{\pi}-1}(\zeta - a_2)^{\frac{\alpha_2}{\pi}-1} \cdots (\zeta - a_n)^{\frac{\alpha_n}{\pi}-1} \tag{55}$$

or, $$z = A\int(\zeta - a_1)^{\frac{\alpha_1}{\pi}-1}(\zeta - a_2)^{\frac{\alpha_2}{\pi}-1} \cdots (\zeta - a_n)^{\frac{\alpha_n}{\pi}-1}\, d\zeta + B \tag{56}$$

where A and B are constants which may be complex, $\alpha_1, \alpha_2, ..., \alpha_n$ are the interior angles of the polygon and $a_1, a_2, ..., a_n$ are the points on the real axis $\eta = 0$ that correspond to the angular points of the polygon in the z-plane.

The following facts should be noted :

1. Any three points of $a_1, a_2, ..., a_n$ can be chosen at will.
2. The constants A and B determine the size, orientation and position of the polygon.
3. It is convenient to choose one point, say a_n at infinity in which case the last factor of (54) and (55) involving a_n is not present.
4. Infinite open polygons can be considered as limiting case of closed polygons.

1.13 Elliptic Coordinates

Let

$$z = c \cosh \zeta, \text{ where } z = x + iy,\ \zeta = \xi + i\eta.$$

Then $x + iy = c \cosh(\xi + i\eta) = e(\cosh \xi \cos \eta + i \sinh \xi \sin \eta)$

so that $x = c \cosh \xi \cos \eta$ $y = c \sinh \xi \sin \eta$. (56)

Obviously $$\frac{x^2}{c^2 \cosh^2 \xi} + \frac{y^2}{c^2 \sinh^2 \xi} = 1 \tag{57}$$

and $$\frac{x^2}{c^2 \cos^2 \eta} + \frac{y^2}{c^2 \sin^2 \eta} = 1. \tag{58}$$

Thus ξ = const. and η = const. represent confocal ellipses and hyperbolas respectively, the distance between the focii being 2c.

Let a, b be the semi-axes of the ellipse (57). Then for $\xi = \alpha$,

$$a = c \cosh \alpha,\ b = c \sinh \alpha,\ c^2 = a^2 - b^2$$

and
$$a + b = ce^{\alpha},\ a - b = ce^{-\alpha},\ \alpha = \frac{1}{2}\log\frac{a+b}{a-b}.$$

The parameters ξ, η are called **elliptic coordinates.**

1.14 The Joukowski Transformations

The transformation

$$z = \zeta + \frac{c^2}{4\zeta} \tag{59}$$

is one of the simplest and most important transformations of two-dimensional motion. By means of this transformation we can map the ζ-plane on the z-plane, and vice versa. From (59), it can be shown that when $|z|$ is large, we have $\zeta = z$ nearly, so that the distant parts of the two-planes are unaltered. Thus a uniform stream at infinity in the z-plane will correspond to a uniform stream of the same strength and direction in the ζ-plane.

We now consider the inverse transformation of (59), viz. $\zeta = \frac{1}{2}\left(z \pm \sqrt{z^2 - c^2}\right)$, or confining to positive sign only,

$$\zeta = \frac{1}{2}\left(z + \sqrt{z^2 - c^2}\right) \tag{60}$$

It can be readily shown that the region outside the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is mapped into the region outside the circle $|\zeta| = \frac{1}{2}(a + b)$.

Application. Streaming past a fixed elliptic cylinder

Let us consider the stream whose complex potential is $U\zeta e^{-i\beta}$ in the ζ-plane. Then, on inserting the circular cylinder $|\zeta| = \frac{1}{2}(a + b)$ into the stream, the new complex potential is given by circle theorem as

$$w_1 = U\zeta e^{-i\beta} + \frac{U(a+b)^2}{4\zeta} e^{i\beta} \tag{61}$$

Now by Joukowski's transformation (60), the region outside the circle $|\zeta| = \frac{1}{2}(a+b)$ is mapped on the region outside the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Hence the complex potential w for the flow past a fixed elliptic cylinder can be obtained from (60) and (61) by eliminating ζ as

$$w = \frac{1}{2}U\left[e^{-i\beta}\left(z+\sqrt{z^2-c^2}\right) + \frac{(a+b)^2 e^{i\beta}}{z+\sqrt{z^2-c^2}}\right]$$

Using the transformation $z = c\cosh\zeta$ for elliptic coordinates, we have $\sqrt{z^2-c^2} = c\sinh\zeta$ and so $z+\sqrt{z^2-c^2} = ce^{\zeta}$. Thus

$$w = \frac{1}{2}U\left[e^{-i\beta}\cdot ce^{\zeta} + \frac{(a+b)^2}{ce^{\zeta}} e^{i\beta}\right]$$

$$= \frac{1}{2}U\sqrt{\frac{a+b}{a-b}}\left[(a-b)e^{\zeta-i\beta} + (a+b)e^{-\zeta+i\beta}\right]$$

Hence on the ellipse $\xi = \alpha$, whence $a + b = ce^{\alpha}$ and $a - b = ce^{-\alpha}$, we get

$$w = \frac{1}{2}U(a+b)\left[e^{\zeta-i\beta-\alpha} + e^{(\zeta-i\beta-\alpha)}\right]$$

i.e. $$w = U(a+b)\cos h(\zeta - i\beta - \alpha). \tag{62}$$

This is the required complex potential for the streaming past a fixed elliptic cylinder.

In particular, if the stream were parallel to the real axis, so that $\beta = 0$, then

$$w = U(a+b)\cos h(\zeta - \alpha). \tag{63}$$

As a special case, we impart to the whole system a velocity U inclined at an angle β with the x-axis. Then the stream is reduced to rest and the cylinder moves with velocity U, so that the complex potential is

$$w = \frac{U(a+b)^2}{4\zeta} e^{i\beta} = \frac{U(a+b)^2}{2(z+\sqrt{z^2-c^2})} e^{i\beta} = \frac{U(a+b)^2}{2c} e^{-\zeta+i\beta}$$

$$= \frac{U(a+b)}{2} e^{-\zeta+i\beta+\alpha}. \tag{64}$$

This is the complex potential for the elliptic cylinder moving in an infinite liquid with velocity U inclined at an angle β with the x-axis. In particular, if the elliptic cylinder moves parallel to the x-axis, so that β = 0, then

$$w = \frac{1}{2}U(a+b)e^{-\zeta+\alpha}. \tag{65}$$

1.15 The Aerofoil

The aerofoil used in modern aeroplanes has a profile of "fish" type, indicated in figure. Such an aerofoil has a blunt leading edge and a sharp trailing edge. The projection of the profile on the double tangent, as shown in the diagram, is the chord. The ratio of the span to the chord is the aspect ratio.

The camber line of a profile is the locus of the point midway between the points in which an ordinate perpendicular to the chord meets the profile. See figure 2.15

The camber is the ratio of the maximum ordinate of the flow round such an aerofoil on the following assumptions :

1. That the air behaves as an incompressible inviscid fluid.
2. That the aerofoil is a cylinder whose cross-section is a curve of the above type.
3. That the flow is two-dimensional irrotational cyclic motion.

The above assumptions are of course only approximations to the actual state of affairs, but by making these simplifications it is possible to arrive at a general understanding of the principles involved.

It has been found that profiles obtained by conformal transformation of circle by the simple Joukowski transformation make good wing shapes, and that the lift can be calculated from the known flow with respect to a circular cylinder.

1.16 The Theorem of Kutta and Joukowski

Statement : *If an aerofoil of any shape be placed in a uniform wind of speed V, then the resultant thrust on the aerofoil is a lift of magnitude* $k\rho v$ *per unit length and is at right angles to the wind, where k is the circulation round the cylinder.*

Direction of flight

Figure 2.15

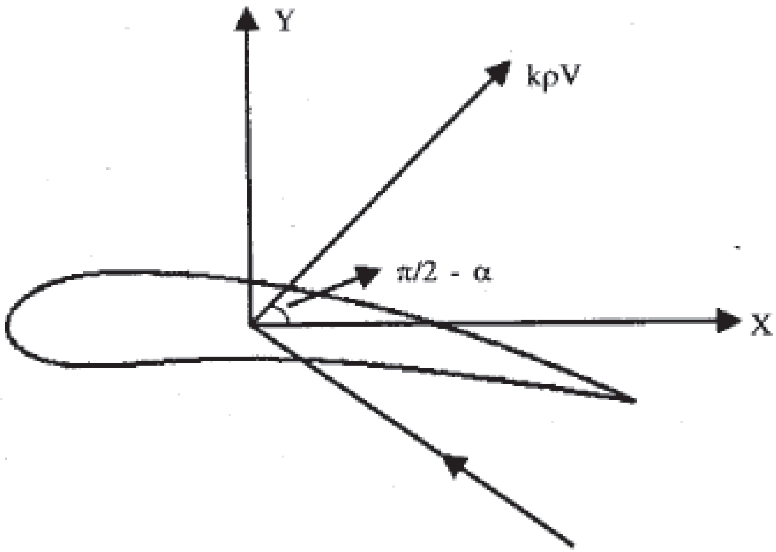

Figure 2.16

Proof. Since there is a uniform wind, the velocity at a great distance from the aerofoil must tend to the wind velocity, and therefore if |z| is sufficiently large, so that we may write

$$-\frac{dw}{dz} = -Ve^{i\alpha} + \frac{A}{z} + \frac{B}{z^2} + \cdots \qquad (66)$$

where α is the angle of incidence or angle of attack.

Thus

$$w = Vze^{i\alpha} - A\ln z + \frac{B}{z} + \cdots$$

and since there is circulation k, we must have

$$-A = \frac{ik}{a\pi}, \qquad (67)$$

for ln z increases by 2πi when we go once round the aerofoil in the positive sense.

From (66) and (67) we get,

$$\left(\frac{dw}{dz}\right)^2 = V^2 e^{2i\alpha} + \frac{ikV}{\pi z} e^{i\alpha} - \frac{k^2 + 8\pi^2 BVe^{i\alpha}}{4\pi^2 z^2} - \cdots. \qquad (68)$$

If we now integrate round a circle whose radius is sufficiently large for the expression (68) to be valid, the theorem of Blasius gives

$$X - iY = \left(\frac{1}{2} i\rho\right) 2\pi i \left(\frac{ikVe^{i\alpha}}{\pi}\right)$$

$$= -ik\rho Ve^{i\alpha}$$

so that, changing the sign of i we obtain

$$X + iY = k\rho Ve^{i\left(\frac{1}{2}\pi - \alpha\right)}.$$

Comparison with above figure shows that this force has all the properties stated in the enunciation.

1.17 Motion of an Elliptic cylinder

(i) To determine the velocity potential and stream function when an elliptic cylinder moves in an infinite liquid with velocity U parallel to the axial plane through the major of a cross-section.

For any cylinder moving with velocities U and V parallel to axes and rotating with an angular velocity ω, we know that on the cylinder

$$\psi = Vx - Uy + \frac{1}{2}\omega(x^2 + y^2) + \text{constant (A, say)}.$$

Here

$$V = 0, \ \omega = 0.$$

Hence the stream function is given by

$$\psi = -Uy + A. \qquad (69)$$

Let the cross-section be the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

This is the same as $\xi = \alpha$, if $a = c\cosh\alpha$, $b = \sinh\alpha$ and $c^2 = a^2 - b^2$, where

$$x = c \cosh \xi \cos \eta, \tag{70}$$

and

$$y = c \sinh \xi \sin \eta. \tag{71}$$

Using (70) and (71), (69) becomes

$$\psi = -Uc \sinh \alpha \sin \eta + A. \tag{72}$$

Since ψ contains $\sin \eta$ and the liquid is at rest at infinity, ψ must be of the form $e^{-\xi} \sin \eta$. We therefore, assume that

$$\phi + i\psi = Be^{-(\xi + i\eta)} \tag{73}$$

so that

$$\psi = -Be^{-\eta} \sin \eta. \tag{74}$$

Then at boundary $\xi = \alpha$, we must have for all values of η,

$$A = 0, B = Uce^{\alpha} \sinh \alpha.$$

Thus

$$\psi = -Uce^{\alpha-\xi} \sinh \alpha \sin \eta \tag{75}$$

is a stream function which will make the boundary of the ellipse a stream line, when the cylinder moves with velocity U.

But

$$c \sinh \alpha = b \text{ and } e^{\alpha} = \left(\frac{a+b}{a-b}\right)^{\frac{1}{2}}. \tag{76}$$

Using (75) and (76), (7) can be written in the form

$$\psi = -Ub\left(\frac{a+b}{a-b}\right)^{\frac{1}{2}} e^{-\xi} \sin \eta. \tag{77}$$

Also from (75),

$$\phi = Ub\left(\frac{a+b}{a-b}\right)^{\frac{1}{2}} e^{-\xi} \cos \eta. \tag{78}$$

Hence we obtain

$$w = \phi + i\psi = Ub\left(\frac{a+b}{a-b}\right)^{\frac{1}{2}} e^{-(\xi+i\eta)}. \tag{79}$$

(ii) To determine the velocity potential and the stream function when an elliptic cylinder moves in an infinite liquid with velocity V parallel to the axial plane through the minor axis of a cross-section.

Proceeding as in (i), we can obtain

$$\phi = Va\left(\frac{a+b}{a-b}\right)^{\frac{1}{2}} e^{-\xi} \cos\eta. \tag{80}$$

$$\psi = Va\left(\frac{a+b}{a-b}\right)^{\frac{1}{2}} e^{-\xi} \sin\eta, \tag{81}$$

and

$$w = iVa\left(\frac{a+b}{a-b}\right)^{\frac{1}{2}} e^{-(\xi+i\eta)}. \tag{82}$$

(iii) To determine the complex potential when an elliptic cylinder moves in an infinite liquid with a velocity v in a direction making an angle β with the major axis of the cross section of the cylinder.

The components of v along coordinate axes are

$$U = v\cos\beta$$

and

$$V = v\sin\beta$$

Let w_1 and w_2 be the complex potentials corresponding to the motion of the cylinder with velocities U and V respectively. Then from (79) and (82), we obtain

$$w_1 = Ub\left(\frac{a+b}{a-b}\right)^{\frac{1}{2}} e^{-(\xi+i\eta)}$$

$$= bv\cos\theta\left(\frac{a+b}{a-b}\right)^{\frac{1}{2}} e^{-(\xi+i\eta)},$$

and

$$w_2 = iva\left(\frac{a+b}{a-b}\right)^{\frac{1}{2}} e^{-(\xi+i\eta)}$$

$$= iav\sin\theta\left(\frac{a+b}{a-b}\right)^{\frac{1}{2}} e^{-(\xi+i\eta)}.$$

Hence the complex potential due to velocity v is given by

$$w = w_1 + w_2$$

$$= cv\left(\frac{a+b}{a-b}\right)^{\frac{1}{2}} e^{-\zeta} \sinh(\alpha + i\beta),$$

where $\zeta = \xi + i\eta$, $b = c \sinh \alpha$, $a = c \cosh \alpha$. Thus

$$w = v(a + b)e^{-\zeta} \sinh(\alpha + i\beta), \text{ since } c^2 = a^2 - b^2.$$

1.18 Liquid Streaming Past a Fixed Elliptic Cylinder

To determine ϕ and ψ for a liquid streaming past a fixed elliptic cylinder with velocity U parallel to major axis of the section.

Superimpose a velocity U on the cylinder and on liquid both in the sense opposite to the velocity of the liquid. This brings the liquid at rest and the cylinder in motion with velocity U. Hence, some suitable term must be added to each of the expressions for ϕ and ψ obtained in (69) of Art. **1.17**. When the stream flows from positive x-axis to negative x-axis, we have

$$-\frac{\partial\phi}{\partial x} = -\frac{\partial\psi}{\partial y} = -U. \tag{83}$$

Accordingly, we must add a term Ux to ϕ and Uy to ψ as obtained in Art. **1.17**. Thus, we have

$$\phi = Ux + Ub\left(\frac{a+b}{a-b}\right)^{\frac{1}{2}} e^{-\xi} \cos\eta$$

$$= U(a^2 - b^2)^{\frac{1}{2}} \cosh\xi\cos\eta + Ub\left(\frac{a+b}{a-b}\right)^{\frac{1}{2}} e^{-\xi} \cos\eta, \tag{84}$$

and

$$\psi = Uy - Ub\left(\frac{a+b}{a-b}\right)^{\frac{1}{2}} e^{-\xi} \sin\eta$$

$$= U(a^2 - b^2)^{\frac{1}{2}} \sinh\xi \sin\eta - Ub\left(\frac{a+b}{a-b}\right)^{\frac{1}{2}} e^{-\xi} \sin\eta. \tag{85}$$

Then the complex potential is given by

$$w = Uz + Ube^{\alpha-\zeta}. \tag{86}$$

Another form for ϕ, ψ and w, we can be obtained as

$$\phi = Uce^{\alpha} \cos\eta \cosh(\xi - \alpha), \tag{87}$$

$$\psi = Uce^{\alpha} \sin\eta \sinh(\xi - \alpha), \tag{88}$$

and

$$\omega = U(a+b)\cosh(\zeta - \alpha). \tag{89}$$

which is the result (63) obtained in Section—1.16.

1.19 Rotating Elliptic Cylinder

To determine ϕ and ψ when an elliptic cylinder is rotating with angular velocity ω in an infinite mass of the liquid at rest at infinity.

For any cylinder moving with velocity U and V parallel to axes and rotating with an angular velocity ω, we know that on the cylinder

$$\psi = Vx - Uy + \frac{1}{2}\omega(x^2 + y^2) + \text{constant, say A}. \tag{90}$$

Let the cross-section be the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

This is the same as $\xi = \alpha$, if $a = c \cosh \alpha$, $b = c \sinh \alpha$ and $c^2 = a^2 - b^2$. The elliptic coordinates (ξ, η) are given by

$$x = c\cosh\xi\cos\eta,$$

$$y = c\sinh\xi\sin\eta. \tag{91}$$

Here

$$U = V = 0.$$

So using (91), (90) reduces to

$$\psi = \frac{1}{4}\omega c^2 (\cosh 2\xi + \cos 2\eta) + A. \qquad (92)$$

Since, ψ contains $\cos 2\eta$ and the liquid is at rest at infinity, ψ must be taken in the form

$$\psi = Be^{-2\xi}\cos 2\eta \qquad (93)$$

and hence

$$\phi = Be^{-2\xi}\sinh 2\eta. \qquad (94)$$

Then at the boundary $\xi = \alpha$, we obtain for all values of η

$$B = \frac{1}{4}\omega c^2 e^{2\alpha}$$

and

$$A = -\frac{1}{4}\omega c^2 \cosh 2\alpha.$$

Thus ϕ and ψ reduce to

$$\phi = \frac{1}{4}\omega (a+b)^2 e^{-2\xi}\sin 2\eta, \qquad (95)$$

$$\psi = \frac{1}{4}\omega (a+b)^2 e^{-2\xi}\cos 2\eta. \qquad (96)$$

Hence the complex potential function is

$$\omega = \frac{1}{4} i\omega (a+b)^2 e^{-2\zeta}, \text{ since } \zeta = \xi + i\eta. \qquad (97)$$

1.20 Motion of a Liquid in Rotating Elliptic Cylinders

Let the elliptic cylinder containing liquid rotate with angular velocity ω. The stream function ψ must satisfy the Laplace's equation

$$\nabla^2\psi = 0$$

and on the boundary it satisfies the condition

$$\psi = \frac{1}{2}\omega (x^2 + y^2) + A. \qquad (98)$$

We assume that

$$\psi = B(x^2 - y^2). \qquad (99)$$

On the boundary of the cylinder, we must have

$$\frac{x^2}{A\Big/\left(B-\frac{1}{2}\omega\right)}+\frac{y^2}{A\Big/\left(-B-\frac{1}{2}\omega\right)}=1. \tag{100}$$

We also know that the boundary of the cylinder is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1. \tag{101}$$

Comparing (100) with (101) we get

$$B=\frac{1}{2}\omega\frac{a^2-b^2}{a^2+b^2}.$$

so that

$$\psi=\frac{1}{2}\omega\frac{a^2-b^2}{a^2+b^2}(x^2-y^2). \tag{102}$$

Then from (99)

$$\phi=-\omega\frac{a^2-b^2}{a^2+b^2}xy. \tag{103}$$

The magnitude of the velocity $\vec{q}$ is given by

$$q^2=\left(-\frac{\partial\phi}{\partial x}\right)^2+\left(-\frac{\partial\phi}{\partial y}\right)^2$$

$$=\omega^2\left(\frac{a^2-b^2}{a^2+b^2}\right)(x^2+y^2). \tag{104}$$

K.E. of the liquid contained in rotating cylinder is given by

$$T=\frac{1}{2}\rho\iint q^2\,dxdy$$

$$=\frac{1}{8}\pi\rho ab\omega^2\frac{(a^2-b^2)^2}{a^2+b^2}. \tag{105}$$

1.21 Flow Past a Plate

If $b=0$, our ellipse degenerates into the line joining the foci, namely $\alpha=0$, and therefor $a=c$. Hence for the flow past a plate inclined at angle θ to the stream, we have

$$\omega = Ua\cosh(\zeta - i\theta)$$

The stagnation points still lie on the hyperbolic branches.

$$\eta = \theta, \; \eta = \pi + \theta.$$

The speed becomes infinite at the edges of the plate, so that the solution cannot represent the complete motion past on an actual plate.

In terms of z, we have

$$\omega = U\left(z\cos\theta - i\sqrt{z^2 - a^2}\sin\theta \right).$$

When the plate is perpendicular to the stream, then $\theta = \frac{\pi}{2}$, so that

$$\omega = -iU\sqrt{z^2 - a^2}.$$

1.22 Illustrative Solved Examples

Example 1 :

In the case of two dimensional motion of a liquid streaming past a fixed circular disc, the velocity at infinity is U in a fixed direction, where U is a variable. Show that the maximum value of the velocity at any point of the fluid is 2U. Prove that the force necessary to hold the disc is $2m\dot{U}$, where m is the liquid displaced by disc.

Solution :

The velocity potential for the liquid streaming past a fixed circular disc is given by

$$\phi = U\left(r + \frac{a^2}{r} \right)\cos\theta, \tag{1}$$

where a is the radius of the disc. This gives

$$\frac{\partial\phi}{\partial r} = U\left(1 - \frac{a^2}{r^2} \right)\cos\theta \text{ and } \frac{\partial\phi}{\partial\theta} = -\left(r + \frac{a^2}{r} \right)\sin\theta$$

Therefore

$$q^2 = \left(-\frac{\partial\phi}{\partial r} \right)^2 + \left(-\frac{1}{r}\frac{\partial\phi}{\partial\theta} \right)^2 = U^2\left(1 - \frac{a^2}{r^2} \right)^2\cos^2\theta + U^2\left(1 + \frac{a^2}{r^2} \right)^2\sin^2\theta$$

$$= U^2\left(1 - \frac{2a^2}{r^2}\cos 2\theta + \frac{a^4}{r^4} \right). \tag{2}$$

which is maximum with respect to θ when $\cos 2\theta = -1$ i.e. $2\theta = \pi$ and then

$$q^2 = U^2\left(1+\frac{2a^2}{r^2}+\frac{a^4}{r^4}\right) \quad \text{or} \quad q = U\left(1+\frac{a^2}{r^2}\right)$$

Now q is further maximum with respect to r when r is minimum, i.e., when r = a. Hence the required maximum value of q is given by

$$q = 2U.$$

By Bernoulli's equation, the pressure p is given by

$$\frac{p}{q} = \frac{\partial\phi}{\partial t} - \frac{1}{2}q^2 + F(t). \qquad (3)$$

Using (1) and (2), (3) reduces to

$$\frac{p}{\rho} = F(t) - \frac{1}{2}U^2\left(1-\frac{2a^2}{r^2}\cos 2\theta+\frac{a^4}{r^4}\right)+\dot{U}\left(r+\frac{a^2}{r}\right)\cos\theta.$$

Putting r = a, the pressure on the boundary of the disc is given by

$$\frac{p}{\rho} = F(t) - 2U^2\sin^2\theta + \dot{U}.2a\cos\theta.$$

Then the resultant pressure on the disc

$$= \int_0^{2\pi}(-p\cos\theta)a\,d\theta = -\rho a\int_0^{2\pi}\left[F(t)-2U^2\sin^2\theta+2\dot{U}a\cos\theta\right]d\theta, \text{ by (4)}$$

$$= -2\rho a^2\dot{U}\int_0^{2\pi}\cos^2\theta\,d\theta = -2\pi a^2\rho\dot{U} = -2m\dot{U} \qquad \text{since } m = \pi a^2\rho$$

Hence the desired force necessary to hold the disc is $2m\dot{U}$.

Example 2 :

A circular cylinder is placed in uniform stream, find the force acting on the cylinder.

Solution :

We know that the complex potential for the undisturbed motion in a uniform stream with velocity components U, V is given by w = (U + iV)z. Using Milne-Thomson's circle theorem, the complex potential for the present problem is

$$w = (U - iV)z + (U + iV)(a^2/z)$$

Therefore

$$\frac{dw}{dz} = U - iV - (U + iV)(a^2/z^2)$$

If the pressure thrusts on the contour of the fixed circular cylinder be represented by a force (X, Y) and a couple of moment N about the origin of co-ordinates, then by Blasius' theorem, we have

$$X - iY = \frac{1}{2} i\rho \int_C \left(\frac{d\omega}{dz} \right)^2 dz = \frac{1}{2} i\rho \int \{ (U - iV) - (U + iV)(a^2 / z^2) \}^2 dz = 0$$

so that

$$X = 0 \quad \text{and} \quad Y = 0 -$$

and

$$N = \text{Real part of } -\frac{1}{2}\rho \int_C z \left(\frac{d\omega}{dz} \right)^2 dz$$

$$= \text{real part of } -\frac{1}{2}\rho \int_C z \left\{ U - iV - (U - iV) \frac{a^2}{z^2} \right\}^2 dz$$

$$= \text{real part of } -\frac{1}{2}\rho \{ -2(U^2 + V^2) a^2 \} 2\pi i = 0$$

Therefore X = Y = N = 0, showing that neither a force nor a couple acts on the cylinder.

Example 3 :

A circular cylinder is fixed across a stream of velocity U with a circulation k round the cylinder. Show that the maximum velocity in the liquid is $2U + (k/2\pi a)$, where a is the radius of the cylinder.

Solution :

The velocity potential ϕ for the motion is

$$\phi = U\left(r + \frac{a^2}{r} \right) \cos\theta - \frac{k\theta}{2\pi}, \qquad (1)$$

where r is measured from the centre of the cross-section of the cylinder.

Then the velocity q is given by

$$q^2 = \left(-\frac{\partial\phi}{\partial r} \right)^2 + \left(-\frac{1}{r}\frac{\partial\phi}{\partial\theta} \right)^2$$

$$= U^2 \left(1 - \frac{a^2}{r^2} \right)^2 \cos^2\theta + \left\{ U\left(1 - \frac{a^2}{r^2} \right) \sin\theta + \frac{k}{2\pi r} \right\}^2$$

$$= U^2\left(1-\frac{2a^2}{r^2}\cos 2\theta+\frac{a^4}{r^4}\right)+\frac{Uk}{\pi r}\left(1+\frac{a^2}{r^2}\right)\sin\theta+\frac{k^2}{4\pi^2 r^2},$$

which is maximum with respect to r when r is minimum, i.e. when r = a.

Thus

$$q^2 = U^2(2-2\cos\theta)+\frac{2Uk}{\pi a}\sin\theta+\frac{k^2}{4\pi^2 a^2}$$

$$= 4U^2\sin^2\theta+\frac{2Uk}{\pi a}\sin\theta+\frac{k^2}{4\pi^2 a^2}$$

$$= \left(2U\sin\theta+\frac{k}{2\pi a}\right)^2 \tag{2}$$

Now q is further maximum with respect to θ when sin θ = 1 i.e. θ = π/2. Thus, from (2) the desired maximum velocity is given by

$$q^2 = \left(2U+\frac{k}{2\pi a}\right)^2 \qquad \text{i.e. } q = 2U+\frac{k}{2\pi a}.$$

Example 4 :

An infinite elliptic cylinder with semi axes a, b is rotating round its axes with angular velocity ω in an infinite liquid of density ρ which is at rest at infinity. Show that if the fluid is under the action of no force, the moment of the fluid pressure on the cylinder round the center is

$$\frac{1}{8}\pi\rho c^4\frac{d\omega}{dt} \quad \text{where} \quad c^2 = a^2 + b^2$$

Solution :

Using Bernoulli's equation, pressure p at any point is given by

$$\frac{p}{q} = C-\frac{1}{2}q^2+\frac{\partial\phi}{\partial t} \tag{1}$$

Now for an elliptic cylinder rotating with an angular velocity ω in an infinite fluid, velocity potential ϕ and complex potential ω are given by

$$\phi = \frac{1}{4}\omega(a+b)^2 e^{-2\xi}\sin 2\eta \tag{2}$$

and

$$w = \frac{1}{4} i\omega (a+b)^2 e^{-2\xi}, \tag{3}$$

where

$$z = x + iy = c \cosh \zeta \quad \text{and} \quad \zeta = \xi + i\eta. \tag{4}$$

Therefore

$$q^2 = \left|\frac{d\omega}{dz}\right|^2 = \left|\frac{d\omega}{d\zeta}\cdot\frac{d\zeta}{dz}\right|^2 = \left|\frac{1}{4} i\omega (a+b)^2 e^{-2\zeta} (-2) \frac{1}{c \sinh \zeta}\right|^2$$

$$= \frac{\omega^2 (a+b)^4}{4c^2} \left|\frac{e^{-2\xi}.e^{-2i\eta}}{\sinh(\xi + i\eta)}\right|^2$$

$$= \frac{\omega^2 (a+b)^4 e^{-4\xi}}{4c^2} \times \frac{1}{\sinh^2 \xi + \sin^2 \eta} \tag{5}$$

and

$$\frac{\partial \phi}{\partial t} = \frac{1}{4}(a+b)^4 e^{-2\xi} \sin 2\eta \frac{d\omega}{dt} \tag{6}$$

Using (1), (5) and (6), the pressure at any point on the boundary of the ellipse $\xi = \alpha$ is given by

$$\frac{p}{\rho} = C - \frac{\omega^2 (a+b)^2 e^{-4\alpha}}{8c^2 (\sinh^2 \alpha + \sin \eta)} + \frac{1}{4}(a+b)^2 e^{-2\alpha} \sin 2\eta \frac{d\omega}{dt} \tag{7}$$

Now the pressure on an elementary are ds of elliptic boundary at a point P (of eccentric angle η) is pds. Let θ be the angle between tangent and radius vector.

Then from calculus, we have

$$\cos\theta = \frac{dr}{ds} \tag{8}$$

Now the moment of the fluid pressure on the element ds about the center

$$= - \text{prds} \cos \theta = - \text{prds}, \qquad \text{by (8)}$$

$$= p.\frac{1}{2}(a^2 + b^2) \sin 2\eta d\eta \qquad \left[\text{since, } rdr = -\frac{1}{2}(a^2 + b^2) \sin 2\eta d\eta\right]$$

Therefore, the required total moment of the liquid pressure on the elliptic cylinder about the centre is

$$=\frac{a^2-b^2}{2}\int_0^{2\pi} p\sin 2\eta\, d\eta$$

$$=\frac{a^2-b^2}{2}\rho\int_0^{2\pi}\left[C-\frac{\omega^2(a+b)^4}{8c^2}\frac{e^{-4\alpha}}{\sinh^2\alpha+\sin^2\eta}+\frac{(a+b)^2}{4}e^{-2\alpha}\sin 2\eta\frac{d\omega}{dt}\right]\sin 2\eta\, d\eta$$

$$=\frac{a^2-b^2}{2}\rho\int_0^{2\pi}\frac{(a+b)^2}{4}e^{-2\alpha}\sin^2 2\eta\frac{d\omega}{dt}\, d\eta \text{ (other integrals vanish)}$$

$$=\frac{(a^2-b^2)(a+b)^2 e^{-2\alpha}}{8}\rho\frac{d\omega}{dt}\int_0^{2\pi}\sin^2 2\eta$$

$$\frac{c^2(a+b)^2}{8}\frac{a-b}{a+b}\rho\frac{d\omega}{dt}\int_0^{2\pi}\frac{1-\cos 4\eta}{2}d\eta \qquad \left[\text{since, } c^2=a^2-b^2,\ e^{2\alpha}=\frac{a+b}{a-b}\right]$$

$$=\frac{c^2(a^2-b^2)}{8}\rho\frac{d\omega}{dt}\pi=\frac{1}{8}\pi\rho c^4\frac{d\omega}{dt}.$$

Example 5 :

In the two-dimensional irrotational motion of a liquid streaming past a fixed elliptic disc $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, the velocity at infinity being parallel to the major axis and equal to U, prove that if

$$x + iy = c\cosh(\xi + i\eta), \quad a^2 - b^2 = c^2 \text{ and } a = c\cosh\alpha,\ b = c\sinh\alpha,$$

the velocity at any point is given by

$$q^2=U^2\frac{a+b}{a-b}\cdot\frac{\sinh^2(\xi-\alpha)+\sin^2\eta}{\sinh^2\xi+\sin^2\eta}$$

and that it has maximum value $\frac{U(a+b)}{a}$ at the end of the minor axis.

Solution :

The velocity potential for the case "Liquid streaming past a fixed elliptic cylinder" is given by

$$w = U(a + b)\cosh(\xi - \alpha) \qquad (1)$$

Now,

$$q = \left|\frac{dw}{dz}\right| = \left|\frac{dw}{d\zeta}\frac{d\zeta}{dz}\right|.$$

Now,

$$q = \frac{U(a+b)}{c}\left|\frac{\sinh(\zeta-\alpha)}{\sinh\zeta}\right| \quad \text{[using (1) and } z = c\cosh\zeta\text{]} \tag{2}$$

But

$$|\sinh(\zeta-\alpha)| = |\sinh(\xi-\alpha+i\eta)|, \quad \text{as } \zeta = \xi + i\eta$$

$$= |\sinh(\xi-\alpha)\cos\eta + i\cosh(\xi-\alpha)\sin\eta|$$

$$= \sqrt{\sinh^2(\xi-\alpha)\cos^2\eta + \cosh^2(\xi-\alpha)\sin^2\eta}$$

$$= \sqrt{\sinh^2(\xi-\alpha) + \sin^2\eta}$$

Similarly,

$$|\sinh\zeta| = \sqrt{\sinh^2\xi + \sin\eta}$$

Since,

$$q = \frac{U(a+b)}{\sqrt{a^2-b^2}}\left[\frac{\sinh^2(\xi-\alpha)+\sin^2\eta}{\sinh^2\xi+\sin^2\eta}\right]^{1/2} \qquad (\because\ c = \sqrt{a^2+b^2})$$

so that

$$q^2 = U^2\left(\frac{a+b}{a-b}\right)\left[\frac{\sinh^2(\xi-\alpha)+\sin^2\eta}{\sinh^2\xi+\sin^2\eta}\right] \tag{3}$$

(3) gives the required value of velocity.

To determine the maximum value of q, we rewrite (3) as follows :

$$q^2 = U^2\left(\frac{a+b}{a-b}\right)\left[1 - \frac{\sinh^2\xi + \sin^2(\xi-\alpha)}{\sinh^2\xi+\sin^2\eta}\right] \tag{4}$$

But sinh ξ > sinh(ξ − α). Hence for a given ξ, (4) shows that q will be maximum when sin η is maximum i.e. $\eta = \frac{\pi}{2}$. Then (3) gives

$$q^2 = U^2\left(\frac{a+b}{a-b}\right)\frac{1+\sinh^2(\xi-\alpha)}{1+\sinh^2\xi} = U^2\left(\frac{a+b}{a-b}\right)\frac{\cosh^2(\xi-\alpha)}{\cosh^2\xi} \qquad (5)$$

$$= U^2\left(\frac{a+b}{a-b}\right)\left[\frac{\cosh\xi\cosh\alpha - \sinh\xi\sin\alpha}{\cosh\xi}\right]^2$$

$$= U^2\left(\frac{a+b}{a-b}\right)(\cosh\alpha - \tanh\xi\sin\alpha)^2, . \qquad (6)$$

showing that q will be maximum when tanh ξ is minimum i.e. ξ is minimum. Since we have an elliptic cylinder surrounded by liquid, the minimum value of ξ is α. Hence putting ξ = α in (5), the required maximum value of q is given by

$$(q_{max})^2 = U^2\left(\frac{a+b}{a-b}\right)\frac{1}{\cosh^2\alpha} = U^2\left(\frac{a+b}{a-b}\right).\frac{c^2}{a^2}, \quad \text{as } a = c\cosh\alpha$$

$$= U^2\left(\frac{a+b}{a-b}\right).\frac{a^2-b^2}{a^2} \qquad \text{as } c = a^2 - b^2$$

Thus

$$(q_{max}) = \left[\frac{U(a+b)}{a}\right]$$

Example 6 :

A source is placed midway between two planes whose distance from one another is 2a. Find the equation of the streamlines when the motion is in two dimensions and show that those particles which at an infinite distance a/2 from one of the boundaries, issued from the source in a direction making an angle π/4 with it.

Solution :

The transformation $\zeta = ie^{\pi z/2a}$ transforms the strip of breadth 2a in the strip of breadth 2a in the z-plane into the upper half of the plane ζ-pane, the origin O′ in the z-pane being midway between the two walls. The points B, C coincide with (B_∞, C_∞), ζ = 0.

When z = 0, ζ = i, i.e., the point P in the ζ-plane.

Thus in the z-pane there is a source m at O′ and equal sink at infinite distance, so in the ζ-pane there will be a source m at P and a sink (−m) at (B, C) and hence an image souuce m at the point ζ = i.

Therefore,

$$w = -m\log(\zeta - i) - m\log(\zeta + i) + m\log\zeta$$

$$= -m\log\frac{\zeta^2+1}{\zeta} = -m\log(\zeta+\zeta^{-1})$$

$$= -m\log\left(ie^{\frac{\pi z}{2a}} - ie^{-\frac{\pi z}{2a}}\right) = -m\log i\left(e^{\frac{\pi z}{2a}} - e^{-\frac{\pi z}{2a}}\right)$$

$$-m\log\left(e^{\frac{\pi z}{2a}} - e^{-\frac{\pi z}{2a}}\right) - m\log i$$

Omitting the constant, we take

$$w = -m\log\left(e^{\frac{\pi z}{2a}} - e^{-\frac{\pi z}{2a}}\right)$$

or

$$w = -m\ \log(e^{cz} - e^{-cz}), \qquad (1)$$

where

$$c = \pi/2a \qquad (2),$$

so that

$$w = -m\ \log\left(e^{c(x+iy)} - e^{-c(x+iy)}\right).$$

Therefore

$$\phi + i\psi = -m\log[2\cos cy\sinh cx + 2i\sin cy\cosh cx]$$

and so

$$\psi = -m\tan^{-1}\frac{2\sin cy\cosh cx}{2\cos cy\sinh cx} = -m\tan^{-1}\left(\frac{\tan cy}{\tanh cx}\right).$$

Streamlines are given by ψ = constant, i.e., tan cy = K tanh cx,

$$\text{i.e.,}\qquad \tan\frac{\pi y}{2a} = K\tanh\frac{\pi x}{2a} \qquad [\text{using } (2)].$$

When $x = \infty$, $y = a/2$. Hence $K = 1$. Therefore streamlines become

$$\tan\frac{\pi y}{2a} = K\tanh\frac{\pi x}{2a}. \tag{3}$$

Diff. (3) w.r.t. x we have

$$\sec^2\frac{\pi y}{2a}\cdot\frac{dy}{dx} = \frac{1}{\cosh^2\frac{\pi x}{2a}}$$

i.e.,

$$\frac{dy}{dx} = \frac{\cos^2\frac{\pi y}{2a}}{\cosh^2\frac{\pi x}{2a}}.$$

Example 7 :

Use the transformation $\zeta = e^{\pi z/a}$ to find the streamlines of the motion in two dimensions due to a source midway between two infinite parallel boundaries (assume the liquid drawn off equally by sinks at the ends of the region). If the pressure tends to zero at the ends of the streams, prove that planes are pressed apart with a force which varies inversely as their distance from each other.

Solution :

We know that the transformation

$$\zeta = e^{\pi z/a} \tag{1}$$

transform the infinite strip $A_\infty, B_\infty, C_\infty, D_\infty$ in the z-plane with origin at O into the upper half in the ζ-plane with origin at (B, C) which coincide with B_∞, C_∞ at $\zeta = 0$. The point $z = ai/2$ goes to $\zeta = e^{\pi i/2} = i$ at the point P in ζ-plane. There is a source at O′ in the z-plane and equal sink at infinity, therefore in the ζ-plane there is a source of strength m at P, sink of strength $(-m)$ at (B, C) and an image source at $\zeta = -i$.

The complex potential is given by

$$w = -m\log(\zeta - i) - m\log(\zeta + i) + m\log\zeta = -m\log(\zeta + \zeta^{-1})$$

$$= -m\log(e^{\pi z/a} + e^{-\pi z/a}), \quad \text{using (1)}$$

$$= -m\log 2 - m\log\cosh(\pi z/a).$$

Therefore

$$w = -m \log \cosh(\pi z / a), \quad \text{omitting the constant term in } \omega.$$

From (2),

$$q = \frac{dw}{dz} = -\frac{m\pi}{a}\tanh\frac{\pi z}{a}, \qquad \text{and } q_\infty = \frac{m\pi}{a}.$$

We know that

$$\frac{p}{\rho} + \frac{1}{2}q^2 = \text{constant} = \frac{1}{2}q_\infty^2, \qquad [p_\infty = 0]$$

i.e.,
$$\frac{p}{\rho} = \frac{\pi^2 m^2}{2a^2}\left(1 - \tanh^2\frac{\pi z}{a}\right) = \frac{\pi^2 m^2}{a^2}\frac{1}{\cosh^2\frac{\pi z}{a}} \tag{3}$$

Now, any point on the upper boundary is z = x + ia and hence (3) gives

$$\frac{p}{\rho} = \frac{\pi^2 m^2}{2a^2}\frac{1}{\cosh^2\left(\frac{\pi x}{a} + i\pi\right)} = \frac{\pi^2 m^2}{2a^2}\frac{1}{\cosh^2\frac{\pi x}{a}}.$$

If F be the force with which the planes are pressed apart, then we have

$$F = 2\int_0^\infty p\,dx = \frac{\pi^2 \rho m^2}{a^2}\int_0^\infty \frac{1}{\cosh^2\frac{\pi x}{a}}dx = \frac{\pi^2\rho m^2}{a^2}\cdot\frac{a}{\pi}\left[\tanh\frac{\pi x}{a}\right]_0^\infty = \frac{\pi\rho m^2}{a},$$

showing that $F \propto \frac{1}{a}$ i.e. the force varies inversely as the distance between the planes apart.

1.23 Model Questions

Short Questions :

1. Show that the curves of equivelocity potential and stream lines intersect orthogonally.
2. Define stream function (or current function).
3. State the boundary conditions for the motion of a cylinder in a uniform stream.

4. Define flow and circulation for fluid motion.

5. Find the expression for the complex velocity potential in the case of motion of a fluid with circulation about a circular cylinder.

6. State Milne-Thomson Circle theorem, Blasius theorem and Kutta-Joukowski theorem.

7. What is meant by conformal mapping? When is it said to be isogonal?

8. Define Schwarz-Christoffel and Joukowski transformations.

9. What is meant by aerofoil? Define camber stating the assumptions required.

10. Define elliptic coordinates.

Broad Questions :

1. Discuss the motion of a circular (or/elliptic) cylinder moving in or infinite mass of the liquid at rest at infinity with velocity U in the direction of x-axis.

2. Discuss the motion of a liquid past a fixed circular (or elliptic) cylinder.

3. Show that if there is a streaming past a fixed circular (or elliptic) cylinder with velocity U in the negative direction of x-axis and there is a circulation of strength k, then the cylinder experiences an upward lift amounting ρkU, ρ being the density of the liquid.

4. Deduce the equation of motion of a circular cylinder moving in a liquid at rest at infinity. Hence show that the effect of the presence of the liquid is to reduce the extraneous force in the ratio $(\sigma - \rho) : (\sigma - \rho)$ where σ, ρ are the densities of the cylinder and liquid respectively.

5. Determine the velocity potential and the stream function at any point of a liquid contained between two coaxial circular cylinders.

6. State and prove Milne-Thomson circle theorem. Apply the theorem to find the complex potential of (i) a uniform flow with velocity U along negative x-axis past a fixed circular cylinder and (ii) a uniform stream at incidence β with positive x-axis.

7. State and prove Blasius theorem and the theorem of Kutta-Joukowski.

8. Determine the complex potential when an elliptic cylinder moves in an infinite liquid with a velocity v in a direction making an angle β with the major axis of the cross-section of the cylinder.

9. Find the complex potential when an elliptic cylinder is rotating with constant angular velocity in an infinite mass of liquid at rest at infinity.

Problems :

1. Show that when a cylinder moves uniformly in a given straight line in an infinite liquid, the path of any point in the fluid is given by the equations

$$\frac{dz}{dt} = \frac{Va^2}{(z' - Vt)^2}; \frac{dz'}{dt} = \frac{Va^2}{(z - Vt)^2},$$

where v = velocity of cylinder, a its radius, and z, z' are x + iy, x − iy and x, y are the coordinates measured from the starting point of the axis, along and perpendicular to its direction of motion.

2. If a long circular cylinder of radius a moves in a straight line at right angles to its length in liquid at rest at infinity, show that when a particle of liquid in the plane of symmetry, initially at distance b in advance of the axis of the cylinder has moved through a distance c, then the cylinder has moved through a distance

$$c + \frac{b^2 - a^2}{b + a\coth(c/a)}.$$

3. A circular cylinder of radius a and infinite length lies on a plane in an infinite depth of liquid. The velocity of liquid at a great distance from the cylinder is U perpendicular to the generators, and the motion is irrotational and two-dimensional. Verify that the stream function is the imaginary part of $w = \pi aU \coth(\pi a/z)$, where z is a complex variable, zero on the line of contact and real on the plane. Prove that the pressure at the two ends of the diameter of the cylinder normal to the plane differs by

$$(1/32)\pi^4\rho U^2.$$

4. The space between two infinitely long cylinders of radii a and b(a > b) respectively is filled with homogenous liquid of density ρ and is suddenly moved with velocity U perpendicular to the axis, the outer one is being kept

fixed. Show that the resultant impulsive pressure on a length l of the inner cylinder is

$$\pi\rho a^2 l \frac{b^2 + a^2}{b^2 - a^2} U.$$

5. Prove that if 2a, 2b are axes of the cross-section of an elliptic cylinder placed across a stream in which the velocity at infinity is U parallel to the major axis of the cross-section, the velocity at a point ($a \cos \eta$, $b \sin \eta$) on the surface is

$$\frac{U(a+b)\sin\eta}{(b^2 \cos^2 \eta + a^2 \sin^2 \eta)^{1/2}}$$

and that, in consequence of the motion, the resultant thrust per unit length on that half of the cylinder on which the stream impinges is diminished by

$$\frac{2b^2 \rho U^2}{a-b}\left[1-\left(\frac{a+b}{a-b}\right)^{1/2} \tan^{-1} A\left(\frac{a-b}{a+b}\right)^{1/2}\right],$$

where ρ is the density of the liquid.

6. An elliptic cylinder, the semi-axes of whose cross-sections are a and b, is moving with velocity U parallel to the major axis of the cross-section, through an infinite liquid of density ρ which is at rest at infinity, the pressure there being Π. Prove that in order that the pressure may everywhere be positive

$$\rho U^2 < \frac{2a^2\Pi}{2ab + b^2}.$$

7. An elliptic cylinder, semi-axes a and b, is held with its length perpendicular to, and its major axis making an angle θ with the direction of a stream of velocity V. Prove that the magnitude of the couple per unit length on the cylinder due to the fluid pressure is

$$\Pi\rho(a^2 - b^2)V^2 \sin\theta\cos\theta$$

and determine its sense.

8. A rectangle open at infinity in the x-direction has solid boundaries along $x = 0$, $y = 0$ and $y = a$. Fluid of amount $2\pi m$ flows into and out of the rectangle at the corners $x = 0, y = 0$ and $x = 0, y = a$ respectively. Prove that the motion of the fluid is given by

$$\omega = 4m \log \tanh (\pi z/2a).$$

9. Show that the transformation $z = (a/\pi)\left\{\sqrt{(\zeta^2 - 1)} - \sec^{-1}\zeta\right\}, \zeta = e^{\pi\omega/aV}$ where $z = x + iy$, $\theta = \phi + i\psi$, give the flow of a straight river of breadth a, running with velocity V at right angles to the straight shore of an otherwise unlimited sea of water into which it flows.

1.24 Summary

In this chapter, two-dimensional irrotational motion of an inviscid liquid past circular an elliptic cylinder has been considered. In addition, motion of these cylinders in the liquid has also been taken into account. Due to wide applications, Milne-Thomson circle theorem and Blasius theorem are discussed. Also a sketch of aerofoil is given.

Unit 2 □ Irrotational Motion in Three-dimensions

Structure

2.0 Introduction

We now describe irrotational motion in three dimensions with particular reference to the motion of a sphere, ellipsoid and solids of revolution in an infinite inviscid incompressible fluid. The stream function and velocity potential are obtained. It is to be noted that the powerful tool of the theory of complex functions cannot used in three dimensional problems.

2.1 Motion of a Sphere

We propose to study irrotational motion in three-dimensions with reference to the motion of a sphere. We shall consider spherical form of solution of the Laplace's equation

$$\frac{\partial^2 \phi}{\partial x^2}+\frac{\partial^2 \phi}{\partial y^2}+\frac{\partial^2 \phi}{\partial z^2}=0 \tag{1}$$

which, in spherical polar co-ordinates (r, θ, ω), reduces to

$$\frac{\partial^2 \phi}{\partial r^2}+\frac{2}{r}\frac{\partial \phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}+\frac{\cot\theta}{r^2}\frac{\partial \phi}{\partial \theta}+\frac{1}{r^2 \sin^2\theta}\frac{\partial^2 \phi}{\partial \omega^2}=0. \tag{2}$$

When there is symmetry about z-axis, ϕ is independent of ω and hence (2) reduces to

$$\frac{\partial^2 \phi}{\partial r^2}+\frac{2}{r}\frac{\partial \phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}+\frac{\cot\theta}{r^2}\frac{\partial \phi}{\partial \theta}=0. \tag{3}$$

Substituting $\phi = f(r)\cos\theta$ in (3), we see that

$$\left(\frac{d^2 f}{dr^2}+\frac{2}{r}\frac{df}{dr}\right)\cos\theta-\frac{f(r)}{r^2}\cos\theta-\frac{\cos\theta}{r^2}f(r)=0,$$

so that f(r) satisfies

$$r^2\frac{d^2 f}{dr^2}+2r\frac{df}{dr}-2f(r)=0$$

which is a homogenous ordinary differential equation and the solution of the equation is of the form $f(r)=Ar+\frac{B}{r^2}$.

Hence the solution of the equation (3) can be taken as

$$\phi=f(r)\cos\theta=\left(Ar+\frac{B}{r^2}\right)\cos\theta. \tag{4}$$

2.1.1 Irrotational motion of liquid in which the sphere is moving :

Let a solid sphere of radius a is moving with velocity U through a homogeneous liquid which is at rest at infinity. Let O, the center of the sphere, be taken as the origin. We choose Oz in the direction of velocity U so that the motion is symmetrical about Oz. Let P(r, θ, ω) be any point, and R' denote the region $r \geq a$ while R is the region $r \leq a$. S(r = a) is the sphere which separates R and R'. If the motion is irrotational then the velocity can be expressed as $\vec{q} = -\vec{\nabla}\phi$, ϕ being the velocity potential. Thus the equation of continuity $\vec{\nabla} \cdot \vec{q} = 0$ gives

$$\nabla^2\phi = 0 \qquad \text{in } R'$$

Since there is symmetry about the z-axis, ϕ is independent of ω and so $\nabla^2\phi = 0$ reduces to

$$\text{i.e. } \frac{\partial^2 \phi}{\partial r^2} + \frac{2}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial \phi}{\partial \theta} = 0. \quad \text{in } R'. \tag{5}$$

Boundary conditions are as follows :

(i) As the liquid is at rest at infinity, we must have

$$-\frac{\partial \phi}{\partial r} = 0 \quad \text{as } r \to \infty, \tag{6}$$

(ii) and as the normal velocity on the sphere is U cos θ, we must have

$$-\frac{\partial \phi}{\partial r} = U\cos\theta \quad \text{on } S(r = a). \tag{7}$$

Since ϕ is harmonic and normal derivative is prescribed at the boundary S(r = a), so ϕ is unique except for an additive constant.

The boundary conditions (i) and (ii) suggest that ϕ must be of the form f(r) cos θ and hence it is assumed as

$$\phi = \left(Ar + \frac{B}{r^2} \right)\cos\theta. \tag{8}$$

From (8)

$$-\frac{\partial \phi}{\partial r} = -\left(A - \frac{2B}{r^3} \right)\cos\theta. \tag{9}$$

Using (6) we get

$$A \cos\theta = 0 \quad \text{i.e.,} \quad A = 0. \tag{10}$$

Using (7) in (9) we get,

$$U\cos\theta = \frac{2B}{a^3}\cos\theta, \text{ for all values of } \theta,$$

so that

$$B = \frac{Ua^3}{2}. \tag{11}$$

Thus

$$\phi = \frac{Ua^3 \cos\theta}{2r^2} \tag{12}$$

which determines the velocity potential for the flow.

We now determine the equation of streamlines of the flow. The differential equation of the streamlines is

$$\frac{dr}{\partial\phi/\partial r} = \frac{rd\theta}{\partial\phi/r\partial\theta}$$

i.e.,

$$\frac{dr}{\frac{Ua^3\cos\theta}{r^3}} = \frac{rd\theta}{\frac{Ua^3\sin\theta}{2r^3}}$$

so that

$$\frac{dr}{r} = \frac{2\cos\theta}{\sin\theta}d\theta.$$

Integrating.

$$\log r = 2\log(\sin\theta) + \log C \quad \text{(C is constant)}$$

i.e., $$r = C\sin^2\theta$$

which is the equation of streamlines.

2.1.2 Equation of motion of a sphere :

We take the origin at the center of the sphere and the z-axis in the direction of motion. Let the sphere move with velocity U along the z-axis in an infinite mass of liquid at rest at infinity. The velocity potential of the motion is given by

$$\phi = \frac{Ua^3}{2r^2}\cos\theta$$

so that

$$\frac{\partial\phi}{\partial r} = -\frac{Ua^3}{r^3}\cos\theta .$$

Let P(a, θ, ω) be the spherical polar co-ordinates of any point on the surface of the sphere. Then the elementary area ds at P is adθ.a sin θdω. Again the value of $\phi\dfrac{\partial\phi}{\partial r}$ at P is given by

$$\left(\phi\frac{\partial\phi}{\partial r}\right)_{r=a} = -\frac{U^2 a\cos^2\theta}{2}. \tag{13}$$

The kinetic energy T_1 of the liquid is

$$T_1 = -\frac{\rho}{2}\iint\phi\frac{\partial\phi}{\partial n}\,ds,$$

integrated over the surface of the sphere, ρ being the density of the liquid. Using (13), we obtain

$$T_1 = -\frac{\rho}{2}\int_{\omega=0}^{2\pi}\int_{\theta=0}^{\pi}\left(-\frac{U^2 a\cos^2\theta}{2}\right)\cdot(a^2\sin\theta d\theta d\omega)$$

$$= \frac{1}{4}U^2\rho a^3\left[\int_0^{\pi}\cos^2\theta\sin\theta d\theta\right]\times\left[\int_0^{2\pi}d\omega\right]$$

$$= \frac{\pi\rho a^3 U^3}{3} = \frac{1}{4}\cdot\frac{4}{3}\pi\rho a^3\,.U^2 = \frac{M'U^2}{4} \tag{14}$$

where, $M' = \dfrac{4\pi a^3}{3}\rho$ is the mass of the liquid displaced by sphere.

Let σ be the density of the sphere and M be the mass of the sphere so that

$$M = \frac{4}{3}\pi\sigma a^3 \tag{15}$$

$$\text{and K.E. of the sphere is } T_2 = \frac{1}{2}MU^2. \tag{16}$$

Let T be the total kinetic energy of the liquid and the sphere. Then

$$T = \frac{1}{2}\left(M + \frac{1}{2}M' \right)U^2, \text{ by (2) and (4).} \tag{17}$$

Let Z be the external force parallel to the z-axis (i.e., in the direction of motion of sphere). Then from the principle of energy, we have

Rate of increase of total K.E. = rate at which work is being done

i.e., $$\frac{d}{dt}\left[\frac{1}{2}\left(M + \frac{1}{2}M' \right)U^2 \right] = ZU$$

i.e., $$\left(M + \frac{1}{2}M' \right)U\dot{U} = ZU, \text{ where } \dot{U} = \frac{dU}{dt}$$

i.e., $$M\dot{U} = Z - \frac{1}{2}M'\dot{U}. \tag{18}$$

Let Z' be the external force on the sphere when no liquid is present. Then from hydrostatical considerations, there exists a relation between Z and Z' of the form

$$Z = [(\sigma - \rho)/\sigma]Z' \tag{19}$$

From (18) and (19), we have

$$M\dot{U} + \frac{1}{2}M'\dot{U} = [(\sigma - \rho)/\sigma]Z'$$

i.e., $$\left(M + \frac{1}{2}M' \right)\dot{U} = [(\sigma - \rho)/\sigma]Z'$$

i.e., $$M\dot{U} = \frac{M}{M + \frac{1}{2}M'} \frac{\sigma - \rho}{\sigma} Z'$$

i.e., $$M\dot{U} = \frac{\frac{4\pi\sigma a^3}{3}}{\frac{4\pi\sigma a^3}{3} + \frac{1}{2}\frac{4\pi\sigma a^3}{3}} \frac{\sigma - \rho}{\sigma} Z'$$

i.e., $$M\dot{U} = \frac{\sigma - \rho}{\sigma + \frac{1}{2}\rho} Z'. \tag{20}$$

Equation (20) shows that the whole effect of the presence of the liquid is to reduce the external force in the ratio $(\sigma - \rho) : \left(\sigma + \frac{1}{2}\rho \right)$.

2.1.3 Fixed sphere in a uniform stream :

Let there be a uniform stream of velocity V in the negative direction of z-axis and the sphere be kept fixed. $R'(r \geq a)$ and $R(r \leq a)$ are the two regions separated by the sphere $S(r = a)$. The motion is irrotational and the velocity potential satisfies

$$\nabla^2\phi = 0 \quad \text{in } R'. \tag{21}$$

Boundary conditions are as follows :

(i) As the sphere is fixed, we have

$$\frac{\partial\phi}{\partial r} = 0, \quad \text{on} \quad S(r = a) \tag{22}$$

(ii) the infinity condition gives

$$\phi \sim Vz \quad \text{as} \quad r \to \infty. \tag{23}$$

The boundary condition (ii) suggests

$$\phi = Vz + \phi_1 \tag{24}$$

$$\text{where } \phi_1 \to 0 \quad \text{as} \quad r \to \infty.$$

Equation (24) gives

$$\nabla^2\phi_1 = 0 \quad \text{in } R' \tag{25}$$

and from (24) by using (2) we get

$$\frac{\partial\phi_1}{\partial r} = V\frac{\partial z}{\partial r} = -V\cos\theta \quad \text{on } S. \tag{26}$$

The conditions (25) and (26) suggest that ϕ_1 must be of the form

$$\phi_1 = \left(Ar + \frac{B}{r^2}\right)\cos\theta \tag{27}$$

A, B being constants.

Using the conditions (25) and (26) we get,

$$\phi_1 = \frac{a^3 V}{2r^2}\cos\theta$$

Hence

$$\phi = Vr\cos\theta + \frac{1}{2}\frac{a^3 V}{r^2}\cos\theta.$$

Here $Vr\cos\theta$ is the velocity potential due to the uniform stream and $\frac{a^3 V}{2r^2}\cos\theta$ is the velocity potential due to the presence of sphere.

Now we determine the lines of flow relative to the sphere.

The streamlines are given by the differential equation

$$\frac{dr}{\partial\phi/\partial r} = \frac{rd\theta}{\partial\phi/r\partial\theta}$$

i.e.,
$$\frac{dr}{V\left(1-\frac{a^3}{r^3}\right)\cos\theta} = \frac{rd\theta}{-V\left(1+\frac{a^3}{2r^3}\right)\sin\theta}$$

i.e.,
$$-2\cot\theta d\theta = \frac{2r^3+a^3}{r^3-a^3}\cdot\frac{dr}{r} = \left(\frac{3r^2}{r^3-a^3}-\frac{1}{r}\right)dr.$$

Integrating

$$-2\log\sin\theta = \log(r^3-a^3) - \log r - \log c$$

where log c is integration constant.

i.e.,
$$r^2\sin^2\theta\left(1-\frac{a^3}{r^3}\right) = c.$$

On the surface of the sphere

$$q_r = \left(-\frac{\partial\phi}{\partial r}\right)_{r=a} = 0$$

$$q_\theta = \left(-\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)_{r=a} = \frac{3V\sin\theta}{2}.$$

We note that $q_\theta = 0$ for $\theta = 0, \pi$ and it is minimum for $\theta = \pi/2, 3\pi/2$ and the minimum value is $\frac{3V}{2}$.

Hence $\theta = 0, \pi$ are the stagnation points on $r = a$.

2.1.4 Moving concentric spheres :

Let the region between two concentric spheres of radii a and b(> a) be filled with liquid which is homogenous and incompressible, R be the region between two concentric spheres i.e., R(a < r < b). Impulses $\vec{I}_1$ and $\vec{I}_2$ are applied on the spheres $S_1(r = a)$ and $S_2(r = b)$ respectively in the z direction so that the two spheres start to move with velocities U and V respectively in the positive direction of z-axis. We intend to determine the resulting motion.

Since the motion is irrotational and symmetric about z-direction, the velocity potential ϕ satisfies the equation

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{2}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial \phi}{\partial \theta} = 0. \quad \text{in } R : (a < r < b). \tag{28}$$

The boundary conditions are

$$(i) \quad -\frac{\partial \phi}{\partial r} = U \cos\theta \quad \text{on} \quad S_1(r = a), \tag{29}$$

$$(ii) \quad -\frac{\partial \phi}{\partial r} = V \cos\theta \quad \text{on} \quad S_2(r = b). \tag{30}$$

The boundary conditions (i) and (ii) suggest that ϕ must be of the form

$$\phi = \left(Ar + \frac{B}{r^2} \right)\cos\theta \tag{31}$$

where A and B are constants.

From (31) we get,

$$-\frac{\partial \phi}{\partial r} = -\left(A - \frac{2B}{r^3} \right)\cos\theta. \tag{32}$$

Using (29) and (30) in (32) we get

$$A = \frac{Ua^3 - Vb^3}{b^3 - a^3} \text{ and } B = \frac{(U-V)a^3b^3}{2(b^3 - a^3)}$$

Therefore, for the starting motion, the velocity potential is given by

$$\phi = \frac{1}{b^3 - a^3}\left[(a^3U - b^3V)r + \frac{a^3b^3(U-V)}{2r^2} \right]\cos\theta. \tag{33}$$

In this case, the impulsive pressures on the boundaries when the motion is started from rest, are $\rho\phi$ so that these are given by

$$\varpi_1 = \frac{a\cos\theta}{b^3 - a^3}\left[\left(a^3 + \frac{b^3}{2} \right)U - \frac{3b^3}{2}V \right]\rho \quad \text{on } S_1$$

and

$$\varpi_2 = \frac{b\cos\theta}{b^3 - a^3}\left[\frac{3a^3}{2}U - \left(a^3 + \frac{b^3}{2}\right)V\right]\rho \quad \text{on } S_2.$$

The impulsive thrust on the inner boundary is therefore,

$$I_1 = \int_0^{\pi} \varpi_1 \cos\theta . 2\pi a^2 \sin\theta d\theta$$

$$= \frac{4\pi a^3 \rho}{3}\left[\left(a^3 + \frac{b^3}{2}\right)U - \frac{3b^3}{2}V\right] / (b^3 - a^3).$$

Similarly, on the outer boundary the impulsive thrust is

$$I_2 = \frac{4\pi b^3 \rho}{3}\left[\frac{3a^2}{2}U - \left(\frac{a^3}{2} + b^3\right)V\right] / (b^3 - a^3).$$

2.2 Axi-symmetric Motion

A motion is called axi-symmetric if it is symmetric about a line, called the axis. Here the motion is the same in every plane through the axis and the plane is called the meridian plane. Now taking the axis of symmetry as z-axis and using the cylindrical coordinate system, every field variable is a function of $\varpi (= (y^2 + x^2)^{1/2})$ and z only.

2.2.1 Stokes' stream function :

Let the axis of symmetry be the axis of z and let $\varpi (= (y^2 + x^2)^{1/2})$ denote distance from the axis. Let u, v denote the components of velocity in the direction of the z and ϖ. Then the equation of continuity is obtained by equating to zero the flow out of the annular space obtained by revolving a small rectangle $d\varpi dz$ around the axis. The total flow out parallel to z is $\frac{\partial}{\partial z}(u 2\pi\varpi d\varpi)dz$ and parallel to ϖ, the total flow out is $\frac{\partial}{\partial \omega}(v.2\pi\varpi dz)d\omega$, so that by equating the sum to zero we get the equation of continuity as

$$\frac{\partial}{\partial z}(u\varpi) + \frac{\partial}{\partial \varpi}(v\varpi) = 0.$$

This is, however, the condition that $v\varpi dz - u\varpi d\varpi$ may be an exact differential, and if we denote this by $d\psi$, we get

$$u = -\frac{1}{\varpi}\frac{\partial\psi}{\partial\varpi},\ v = \frac{1}{\varpi}\frac{\partial\psi}{\partial z}$$

This function ψ is called Stokes' stream function.

The streamlines are given by

$$\frac{dz}{u} = \frac{d\varpi}{v}$$

$$\text{i.e., } \varpi(v dz - u d\varpi) = 0,$$

that is, by $d\psi = 0$. Hence the equation ψ = constant represents stream lines.

A property of Stokes' stream function is that 2π times the difference of its values at two points in the same meridian plane is equal to the flow across the annular surface obtained by the revolution round the axis joining the points. For, if ds be an element of the curve and θ its inclination to its axis, the flow outwards across the surface of revolution is

$$\int (v\cos\theta - u\sin\theta).2\pi\varpi ds = 2\pi\int\left(\frac{\partial\psi}{\partial x}dx + \frac{\partial\psi}{\partial\varpi}d\varpi\right) = 2\pi\int d\psi = 2\pi(\psi_2 - \psi_1).$$

We might also define the value of Stokes' stream function at any point P as $\frac{1}{2\pi}$ of the amount of flow across a surface obtained by revolving a curve AP round the axis, A being a fixed point in the meridian plane through P; for, this makes

$$\psi = \frac{1}{2\pi}\int_A^P (v\cos\theta - u\sin\theta).2\pi\varpi ds$$

$$= \int_A^P (v\varpi dz - u\varpi d\varpi)$$

and by varying the position of P, we get as before,

$$u = -\frac{1}{\varpi}\frac{\partial\psi}{\partial\varpi} \text{ and } v = \frac{1}{\varpi}\frac{\partial\psi}{\partial z}. \quad (34)$$

2.2.2 Irrotational axi-symmetric motion :

Let us consider an irrotational motion for which the velocity potential is ϕ. Therefore,

$$u = -\frac{\partial\phi}{\partial z},\ v = -\frac{\partial\phi}{\partial\varpi}. \quad (35)$$

Again Stokes' stream function always exists such that

$$u = -\frac{1}{\varpi}\frac{\partial\psi}{\partial\varpi} \text{ and } v = \frac{1}{\varpi}\frac{\partial\psi}{\partial z}. \tag{36}$$

Thus

$$\frac{\partial\phi}{\partial z} = \frac{1}{\varpi}\frac{\partial\psi}{\partial\varpi}, \quad \frac{\partial\phi}{\partial\varpi} = -\frac{1}{\varpi}\frac{\partial\psi}{\partial z} \tag{37}$$

From (37)

$$\frac{\partial}{\partial\varpi}\left(\frac{\partial\phi}{\partial z}\right) = \frac{\partial}{\partial\varpi}\left(\frac{1}{\varpi}\frac{\partial\psi}{\partial\varpi}\right)$$

$$\text{and } \frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial\varpi}\right) = -\frac{\partial}{\partial z}\left(\frac{1}{\varpi}\frac{\partial\psi}{\partial z}\right)$$

so that

$$\frac{\partial}{\partial\varpi}\left(\frac{1}{\varpi}\frac{\partial\psi}{\partial\varpi}\right) = -\frac{\partial}{\partial z}\left(\frac{1}{\varpi}\frac{\partial\psi}{\partial z}\right) \qquad \left[\because \frac{\partial^2\phi}{\partial\varpi\partial x} = \frac{\partial^2\phi}{\partial x\partial\varpi}\right]$$

$$\text{i.e.,} \quad -\frac{1}{\varpi^2}\frac{\partial\psi}{\partial\varpi} + \frac{1}{\varpi}\frac{\partial^2\psi}{\partial\varpi^2} = -\frac{1}{\varpi}\frac{\partial^2\psi}{\partial z^2},$$

$$\text{i.e.,} \quad \frac{\partial^2\psi}{\partial z^2} + \frac{\partial^2\psi}{\partial\varpi^2} - \frac{1}{\varpi}\frac{\partial\psi}{\partial\varpi} = 0. \tag{38}$$

Again from (37)

$$\frac{\partial}{\partial z}\left(\frac{\partial\psi}{\partial\varpi}\right) = \frac{\partial}{\partial z}\left(\varpi\frac{\partial\phi}{\partial z}\right).$$

and

$$\frac{\partial}{\partial\varpi}\left(\frac{\partial\psi}{\partial z}\right) = -\frac{\partial}{\partial\varpi}\left(\varpi\frac{\partial\phi}{\partial\varpi}\right)$$

so that

$$\frac{\partial}{\partial z}\left(\varpi\frac{\partial\phi}{\partial z}\right) = -\frac{\partial}{\partial\varpi}\left(\varpi\frac{\partial\phi}{\partial\varpi}\right) \qquad \left[\because \frac{\partial^2\psi}{\partial z\partial\varpi} = \frac{\partial^2\psi}{\partial\varpi\partial z}\right]$$

$$\text{i.e.,}\quad \varpi\frac{\partial^2\phi}{\partial z^2} = -\varpi\frac{\partial^2\phi}{\partial\varpi^2} - \frac{\partial\phi}{\partial\varpi},$$

$$\text{i.e.,}\quad \frac{\partial^2\phi}{\partial z^2} + \frac{\partial^2\phi}{\partial\varpi^2} + \frac{1}{\varpi}\frac{\partial\phi}{\partial\varpi} = 0. \tag{39}$$

Equations (38) and (39) show that ϕ and ψ are not interchangeable in the way that is applied to the velocity potential and stream function of two-dimensional irrotational motion.

Now we rewrite (38) and (39) in polar co-ordinates. Let q_r and q_θ be the velocities in the directions of dr and $rd\theta$. Then, since $\varpi = r\sin\theta$ and the velocity from right to left across ds is $\frac{1}{\varpi}\frac{\partial\psi}{\partial s}$, we get

$$q_r = -\frac{1}{\varpi}\frac{\partial\psi}{r\partial\theta} = -\frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta},$$

$$q_\theta = \frac{1}{\varpi}\frac{\partial\psi}{\partial r} = \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}. \tag{40}$$

But in irrotational motion, we know that

$$q_r = -\frac{\partial\phi}{\partial r},\quad q_\theta = -\frac{\partial\phi}{r\partial\theta} \tag{41}$$

and since $$\frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta} = \frac{\partial\phi}{\partial r} \text{ and } \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r} = -\frac{1}{r}\frac{\partial\phi}{\partial\theta}, \tag{42}$$

so $$\frac{\partial}{\partial\theta}\left(\frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}\right) = \frac{\partial^2\phi}{\partial\theta\partial r} = -\frac{\partial}{\partial r}\left(\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}\right)$$

$$\text{i.e.,}\quad r^2\frac{\partial^2\psi}{\partial r^2} + \sin\theta\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\theta}\right) = 0. \tag{43}$$

Let $\mu = \cos\theta$ so that

$$\sin\theta\frac{\partial}{\partial\mu} = -\frac{\partial}{\partial\theta}, \tag{44}$$

then (43) reduces to

$$r^2 \frac{\partial^2 \psi}{\partial r^2} \sin^2 \theta \frac{\partial}{\partial \mu}\left(\frac{\partial \psi}{\partial \mu}\right) = 0. \tag{45}$$

Similarly eliminating ψ from (42), we get

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial \phi}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \phi}{\partial \theta}\right) = 0$$

$$\text{i.e.,}\quad \frac{\partial}{\partial r}\left(r^2 \frac{\partial \phi}{\partial r}\right) + \frac{\partial}{\partial \mu}\left[(1-\mu^2)\frac{\partial \phi}{\partial \mu}\right] = 0 \tag{46}$$

which is Laplace's equation and has solution of the forms $r^n P_n(\mu)$ and $r^{-n-1}P_n(\mu)$, $P_n(\mu)$ being the Legendre polynomial of degree n.

Again from (42), we have

$$\frac{\partial \psi}{\partial \mu} = -r^2 \frac{\partial \phi}{\partial r} = -nr^{n+1} P_n \quad \text{or} \quad (n+1) r^{-n} P_n, \tag{47}$$

$$\frac{\partial \psi}{\partial r} = (1-\mu^2)\frac{\partial \phi}{\partial \mu} = (1-\mu^2) r^n \frac{\partial P_n}{\partial \mu} \quad \text{or} \quad (1-\mu^2) r^{-n-1} \frac{\partial P_n}{\partial \mu}. \tag{48}$$

On integration, (48) gives us possible solutions for ψ as

$$\psi = \frac{(1-\mu^2)}{n+1} r^{n+1} \frac{\partial P_n}{\partial \mu} \quad \text{or} \quad -\frac{(1-\mu^2)}{n}\frac{1}{r^n}\frac{\partial P_n}{\partial \mu}. \tag{49}$$

2.2.3 Solids of revolution moving along their axes in an infinite mass of liquid :

Suppose that a solid moves along Ox with velocity U and let Ox be the axis of revolution. Since the motion is symmetrical about Ox, Stokes' stream function exists.

Now the normal velocity of the liquid in contact with the surface at P is $-\frac{1}{\varpi}\frac{\partial \psi}{\partial s}$. On the boundary, we have

$$-\frac{1}{\varpi}\frac{\partial \psi}{\partial s} = \text{velocity of the solid along normal}$$

$$\text{i.e., } -\frac{1}{\varpi}\frac{\partial\psi}{\partial s} = U\cos\theta = U\frac{\partial\varpi}{\partial s}, \text{ where } \cos\theta = \frac{\partial\varpi}{\partial s}$$

$$\text{i.e., } d\psi = -U\varpi d\varpi$$

Integrating,

$$\psi = -\frac{U\varpi^2}{2} + \text{constant}$$

$$\text{i.e., } \psi = -\frac{Ur^2 \sin^2\theta}{2} + \text{constant, where } \varpi = r\sin\theta \tag{50}$$

$$\text{i.e., } \psi = -\frac{U(1-\mu^2)}{2} + \text{constant, where } \mu = \cos\theta \tag{51}$$

which is the boundary condition at P.

Again ψ must satisfy the equation

$$r^2\frac{\partial^2\psi}{\partial r^2} + (1-\mu^2)\frac{\partial^2\psi}{\partial\mu^2} = 0, \text{ where } \mu = \cos\theta \tag{52}$$

and it is known that (52) has solutions of the form $\frac{1-\mu^2}{n+1} r^{n+1} \frac{\partial P_n}{\partial\mu}$ and $\frac{1-\mu^2}{nr^n}\frac{\partial P_n}{\partial\mu}$

As an example, we consider the case of a sphere of radius a. Then with $r = a$ in (51), we must have

$$\psi = -\frac{Ua^2}{2}(1-\mu^2) + C \tag{53}$$

Taking $n = 1$ in (49), we have the solution of the form

$$\psi = A\frac{1-\mu^2}{r}, \tag{54}$$

then at the boundary we must have

$$\frac{A(1-\mu^2)}{a} = -\frac{Ua^2}{2}(1-\mu^2) + C$$

for all values of μ. This requires that $C = 0$ and $A = -\frac{Ua^3}{2}$. Hence putting these values and noting that $\mu = \cos\theta$, (54) gives

$$\psi = -\frac{Ua^3\sin^2\theta}{2r} \tag{55}$$

Again we know that

$$(1-\mu^2)\frac{\partial\phi}{\partial\mu} = \frac{\partial\psi}{\partial r} = \frac{Ua^3 \sin^2\theta}{2r^2}$$

$$\text{i.e.,}\ \frac{\partial\phi}{\partial\mu} = \frac{Ua^3}{2r^2}.$$

Integrating

$$\phi = \frac{Ua^3}{2r^2}\mu = \frac{Ua^3}{2r^2}\cos\theta. \tag{56}$$

2.3 Ellipsoidal Coordinate System

Let us consider the equation

$$\frac{x^2}{a^2+\theta} + \frac{y^2}{b^2+\theta} + \frac{z^2}{c^2+\theta} = 1,\ a > b > c \tag{57}$$

where θ is a parameter. This represents a family of confocal central conicoids. The above equation can be reduced to a cubic equation of θ, given by

$$F(\theta) \equiv x^2(b^2+\theta)(c^2+\theta) + y^2(a^2+\theta)(c^2+\theta) + z^2(a^2+\theta)(b^2+\theta)$$
$$-(a^2+\theta)(b^2+\theta)(c^2+\theta) = 0. \tag{58}$$

Now

$$F(-\infty) = +\text{ve},\ F(-a^2) = +\text{ve},\ F(-b^2) = -\text{ve},\ F(-c^2) = +\text{ve},\ F(\infty) = -\text{ve}.$$

Hence we conclude that $F(\theta)$ has three real roots λ, μ, ν such that

$$-a^2 < \nu < -b^2 < \mu < -c^2 < \lambda.$$

Thus through any fixed point (x, y, z), there are three conicoids represented by

λ = constant, μ = constant, ν = constant

It may be noted that

λ = constant represents an ellipsoid,

μ = constant represents a hyperboloid of one sheet,

and

ν = constant represents a hyperboloid of two sheets.

Now we write

$$f(\lambda) \equiv \frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} + \frac{z^2}{c^2+\lambda} - 1 = 0. \tag{58}$$

Differentiating with respect to x we get,

$$f'(\lambda)\frac{\partial\lambda}{\partial x} = \frac{2x}{a^2+\lambda}$$

$$\text{i.e., } \frac{\partial\lambda}{\partial x} = \frac{1}{f'(\lambda)}\frac{2x}{a^2+\lambda}.$$

Similarly

$$\frac{\partial\lambda}{\partial y} = \frac{1}{f'(\lambda)}\frac{2y}{b^2+\lambda},$$

$$\frac{\partial\lambda}{\partial z} = \frac{1}{f'(\lambda)}\frac{2z}{c^2+\lambda}.$$

Direction cosines of the normal to the surface λ = constant are proportional to $\left(\frac{\partial\lambda}{\partial x}, \frac{\partial\lambda}{\partial y}, \frac{\partial\lambda}{\partial z}\right)$. Similarly, direction cosines of the normal to the surface μ = constant are proportional to $\left(\frac{\partial\mu}{\partial x}, \frac{\partial\mu}{\partial y}, \frac{\partial\mu}{\partial z}\right)$. Now the cosine of the angle between these normals is proportional to

$$\frac{\partial\lambda}{\partial x}\frac{\partial\mu}{\partial x} + \frac{\partial\lambda}{\partial y}\frac{\partial\mu}{\partial y} + \frac{\partial\lambda}{\partial z}\frac{\partial\mu}{\partial z}$$

$$= \frac{1}{f'(\lambda)f'(\mu)}\left[\frac{4x^2}{(a^2+\lambda)(a^2+\mu)} + \frac{4y^2}{(b^2+\lambda)(b^2+\mu)} + z\right]$$

$$= \frac{4}{f'(\lambda)f'(\mu)}(f(\lambda) - f(\mu)) \tag{59}$$

which vanishes if $f(\lambda) = 0$, $f(\mu) = 0$. Hence λ, μ, ν give the system of orthogonal curvilinear coordinates called ellipsoidal co-ordinates. Again λ, μ, ν are the roots of $F(\theta) = 0$, so that $F(\theta)$ can be written as

$$F(\theta) = (\lambda - \theta)(\mu - \theta)(\nu - \theta).$$

Let us put $\theta = -a^2, -b^2, -c^2$ in (35) successively and we get

$$x^2 = \frac{(a^2+\lambda)(a^2+\mu)(a^2+\nu)}{(a^2-b^2)(a^2-c^2)}$$

$$y^2 = \frac{(b^2+\lambda)(b^2+\mu)(b^2+\nu)}{(b^2-a^2)(b^2-c^2)}$$

$$z^2 = \frac{(c^2+\lambda)(c^2+\mu)(c^2+\nu)}{(c^2-a^2)(c^2-b^2)}$$

Now if ds is an element then

$$ds^2 = h_1^2 d\lambda^2 + h_2^2 d\mu^2 + h_3^2 d\nu^2$$

where

$$h_1^2 = \left(\frac{\partial x}{\partial \lambda}\right)^2 + \left(\frac{\partial y}{\partial \lambda}\right)^2 + \left(\frac{\partial z}{\partial \lambda}\right)^2,$$

$$h_2^2 = \left(\frac{\partial x}{\partial \mu}\right)^2 + \left(\frac{\partial y}{\partial \mu}\right)^2 + \left(\frac{\partial z}{\partial \mu}\right)^2,$$

$$h_3^2 = \left(\frac{\partial x}{\partial \nu}\right)^2 + \left(\frac{\partial y}{\partial \nu}\right)^2 + \left(\frac{\partial z}{\partial \nu}\right)^2.$$

Now it is easy to see that

$$4h_1^2 = \frac{x^2}{(a^2+\lambda)^2} + \frac{y^2}{(b^2+\lambda)^2} + \frac{z^2}{(c^2+\lambda)^2};$$

similarly,

$$4h_2^2 = \frac{x^2}{(a^2+\mu)^2} + \frac{y^2}{(b^2+\mu)^2} + \frac{z^2}{(c^2+\mu)^2},$$

$$4h_3^2 = \frac{x^2}{(a^2+\nu)^2} + \frac{y^2}{(b^2+\nu)^2} + \frac{z^2}{(c^2+\nu)^2}.$$

We can write

$$f(\theta) = \frac{(\lambda-\theta)(\mu-\theta)(\nu-\theta)}{(a^2+\theta)(b^2+\theta)(c^2+\theta)},$$

then

$$-f'(\lambda) = \frac{(\lambda-\mu)(\lambda-\nu)}{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}.$$

Thus

$$4h_1^2 = \frac{(\lambda-\mu)(\lambda-\nu)}{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)},$$

$$4h_2^2 = \frac{(\mu-\lambda)(\mu-\nu)}{(a^2+\mu)(b^2+\mu)(c^2+\mu)},$$

$$4h_3^2 = \frac{(\nu-\lambda)(\nu-\mu)}{(a^2+\nu)(b^2+\nu)(c^2+\nu)}.$$

So the Laplace operator in ellipsoidal coordinates is

$$\nabla^2\phi = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial\lambda}\left(\frac{h_2h_3}{h_1}\frac{\partial\phi}{\partial\lambda}\right)+\frac{\partial}{\partial\mu}\left(\frac{h_3h_1}{h_2}\frac{\partial\phi}{\partial\mu}\right)+\frac{\partial}{\partial\nu}\left(\frac{h_1h_2}{h_3}\frac{\partial\phi}{\partial\nu}\right)\right]$$

$$=(\mu-\nu)\left(K_\lambda\frac{\partial}{\partial\lambda}\right)^2\phi+(\nu-\lambda)\left(K_\mu\frac{\partial}{\partial\mu}\right)^2\phi+(\lambda-\mu)\left(K_\nu\frac{\partial}{\partial\nu}\right)^2\phi$$

where

$$K_\lambda = (a^2-\lambda)(b^2-\lambda)(c^2-\lambda),$$

$$K_\mu = (a^2+\mu)(b^2+\mu)(c^2+\mu),$$

$$K_\nu = (a^2+\nu)(b^2+\nu)(c^2+\nu),$$

Solutions of this Laplace equation are called ellipsoidal harmonics.

2.3.1 Translatory motion of an ellipsoid :

We consider the ellipsoid $S : \lambda = 0$,

$$\text{i.e.,}\quad \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=0. \qquad (60)$$

which moves through a liquid in the direction of x-axis with velocity U. Since the motion is irrotational, the velocity potential ϕ satisfies

$$\nabla^2\phi = 0 \quad \text{for} \quad \lambda \geq 0.$$

The boundary conditions are

$$\text{(i)} \; -\frac{\partial\phi}{\partial n} = U\cos\theta_x, \text{ on } \lambda = 0$$

where θ_x is the angle between the normal and x-axis,

$$\text{i.e.,} \; -\frac{\partial\phi}{\partial\lambda} = -U\frac{\partial x}{\partial\lambda}, \; \lambda = 0,$$

since $dn = h_1 d\lambda$, $\cos\theta_x = \frac{1}{h_1}\frac{\partial x}{\partial\lambda}$. Thus

$$\phi = -Ux \quad \text{on} \quad \lambda = 0. \tag{61}$$

(ii) ϕ is regular at infinity

$$\text{i.e.,} \quad \phi \to 0 \quad \text{as} \quad \lambda \to \infty. \tag{62}$$

For solution of the Laplace equation (60) in the ellipsoidal coordinate system, we take

$$\phi = Cx\int_\lambda^\infty \frac{dt}{(a^2+t)K_t} \text{ which tends to 0 as } \lambda \to \infty \tag{63}$$

where C is constant.

Using the boundary condition (61) in (62) we get

$$-U\frac{\partial x}{\partial\lambda} = C\frac{\partial x}{\partial\lambda}\int_0^\infty \frac{dt}{(a^2+t)K_t} - \frac{Cx}{a^2.abc}, \quad \text{where } \lambda = 0$$

Again

$$\frac{\partial x}{\partial\lambda} = \frac{x}{2a^2}, \quad \text{when } \lambda = 0$$

therefore,

$$C = \frac{abcU}{2-\alpha_0} \text{ where } \alpha_0 = abc\int_0^\infty \frac{dt}{(a^2+t)K_t}. \tag{64}$$

Thus finally we get

$$\phi = \frac{abcUx}{2-\alpha_0}\int_\lambda^\infty \frac{dt}{(a^2+t)^{3/2}(b^2+t)^{1/2}(a^2+t)^{1/2}} \tag{65}$$

and on the ellipsoid we have from (64)

$$\phi = \frac{\alpha_0 xU}{2-\alpha_0}. \tag{66}$$

The kinetic energy of the liquid is

$$T = -\frac{1}{2}\rho\int \phi \frac{\partial\phi}{\partial n} ds = \frac{\alpha_0 \rho U^2}{2(2-\alpha_0)}\int x\cos\theta_x \, ds.$$

Since $\cos\theta_x$ ds is the projection on the plane x = 0 of the area ds of the surface, and the last integral gives the volume of the ellipsoid as $\frac{4\pi abc}{3}$ we find

$$T = \frac{M'\alpha_0 U^2}{2(2-\alpha_0)}$$

where M′ is the mass of liquid displaced by ellipsoid.

When the ellipsoid has, in addition, velocity components V, W parallel to y-axis and z-axis, we get, by superposing the results analogous to (66), the velocity potential to be

$$\frac{abcU_x}{2-\alpha_0}\int_\lambda^\infty \frac{dt}{(a^2+t)K_t} + \frac{abcV_y}{2-\beta_0}\int_\mu^\infty \frac{dt}{(b^2+t)K_t} + \frac{abcW_z}{2-\gamma_0}\int_\nu^\infty \frac{dt}{(c^2+t)K_t}$$

where β_0, γ_0 are defined by writing $b^2 + t$, $c^2 + t$ for $a^2 + t$ in (5).

2.4 Source, Sink, Doublet

Source :

Source is a point at which liquid is created and distributed at a uniform rate and the liquid flows outward symmetrically in all directions from the point. If the rate of emission of the volume of liquid is $4\pi m$, then m is called the *strength of the source*. When the rate of emission is constant then the source is called steady.

Let us consider a steady irrotational motion due to the source of strength m. The volume of the liquid flowing out in a spherical surface of radius r and the source at its center must be equal to the volume of liquid created per unit time. Let ϕ be the velocity potential due to a simple source of strength m, and the liquid be at rest at infinity. Then

$$4\pi m = \text{flux of liquid across the spherical surface} = -\frac{\partial\phi}{\partial r}\cdot 4\pi r^2$$

So,

$$\phi = \frac{m}{r} + \text{constant}$$

Since constant velocity potential does not change the motion, we may neglect the constant or may redefine the velocity potential by including the constant in it.

Sink :

A *sink* is a source of negative strength.

Note : A source or sink implies creation or annihilation of fluid at a point. Both are points at which the velocity potential is infinite. A source and sink are purely abstract conception but they are to be considered due to exigencies of analysis.

Doublet :

A combination of source and sink of equal strength m at a small distance δs apart, when the limit of m is infinitely large and δs is infinitely small, but $m\delta s$ remains finite and equal to μ, then it is called a *doublet* of strength μ and the line δs taken from – m to m is called the axis of the doublet. Let $\vec{\nu}$ denotes the direction of the axis of the doublet. So,

$$[\phi]_P = \lim_{m\delta s \to \mu}\left[-m\left(\frac{1}{r}\right)_Q + m\left(\frac{1}{r}\right)_{Q'}\right] = \lim_{m\delta s \to \mu}\left[m\delta s\,\frac{\left(\frac{1}{r}\right)_{Q'} - \left(\frac{1}{r}\right)_Q}{\delta s}\right] = \mu\,\frac{\partial}{\partial \nu}\left(\frac{1}{r}\right)$$

where the source is at Q and the sink is at Q′, and in the limit both Q and Q′ tend to P. Thus

$$[\phi]_P = \mu\frac{\partial}{\partial \nu}\left(\frac{1}{r}\right) = -\frac{\mu}{r^2}\frac{\partial r}{\partial \nu}.$$

Again, since $r = -\nu\cos\theta$

$$[\phi]_P = -\frac{\mu}{r^2}\frac{\partial}{\partial \nu}(-\nu\cos\theta) = \frac{\mu\cos\theta}{r^2}$$

2.5 Images

If in a liquid a surface S can be drawn across which there is no flow, then any systems of sources, sinks and doublets on opposite sides of this surface may be said to be images of one another with regard to the surface. And if the surface S be regarded as a rigid boundary and the liquid is removed from one side of it, the motion on the other side will remain unaltered.

2.5.1 Image of a source with respect to a rigid plane :

Let S(x = 0) be a fixed plane and a source of strength m be placed at Q(a, 0, 0) in front of S (see figure 1.1.). Let Q′(− a, 0, 0) be another point which is image point of Q with respect to S. Let P be any fixed point and r_1, r_2 be the distances of Q, Q′ respectively from P.

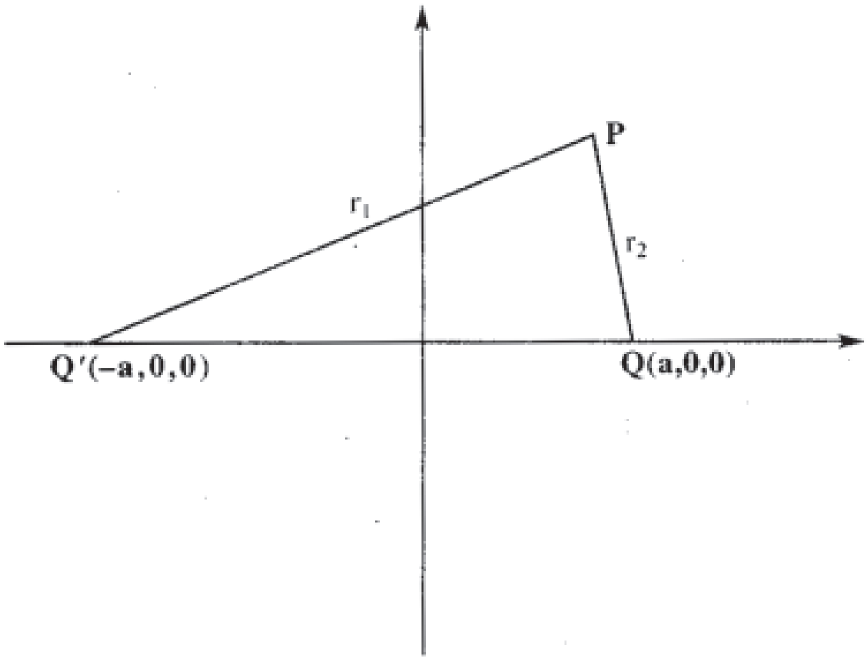

Figure 2.1

Since the motion is irrotational on the right of S(x = 0) due to the source at Q, so

$$\nabla^2\phi = 0 \text{ in } R : x \geq 0 \text{ except at } Q,$$

therefore,

$$\phi \sim \frac{m}{r_1} \text{ near Q}$$

where r_1 is the distance from Q ($r_1 \to 0$), and also ϕ is regular at infinity. Again, $\frac{\partial \phi}{\partial x} = 0$ on $S(x = 0)$.

Now we set

$$\phi = \frac{m}{r_1} + \phi_1$$

where ϕ_1 is due to the presence of the rigid wall. Then

$$\nabla^2 \phi_1 = \nabla^2 \phi - \nabla^2 \left(\frac{m}{r_1} \right) = 0$$

and

$$\frac{\partial \phi_1}{\partial x} = \frac{\partial \phi}{\partial x} - m \frac{\partial}{\partial x} \left(\frac{1}{r_1} \right) = -m \frac{\partial}{\partial x} \left(\frac{1}{r_1} \right) \text{ on S.}$$

Now

$$r_1^2 = (x - a)^2 + y^2 + z^2, \quad r_2^2 = (x + a)^2 + y^2 + z^2$$

$$\frac{\partial \phi_1}{\partial x} = -m \frac{\partial}{\partial x} \left(\frac{1}{r_1} \right) = \frac{ma}{r_1^3} \text{ on S}$$

We choose $\phi_1 = \frac{m}{r_2}$, the reason for this is as follows :

$$\frac{\partial \phi_1}{\partial x} = -m \frac{\partial}{\partial x} \left(\frac{1}{r_2} \right) = \frac{ma}{r_2^3} \text{ on } x = 0.$$

Therefore, on $x = 0$

$$\frac{ma}{r_1^3} = \frac{ma}{r_2^3}, \text{ which is obvious.}$$

Hence

$$\phi_1 = \frac{m}{r_2} \text{ in R.}$$

Therefore,

$$\phi = \frac{m}{r_1} + \frac{m}{r_2}$$

This shows that the image of a point source with respect to a point is a point source of same strength at the image point.

2.5.2 Image of a source in front of a sphere :

Let S(r = a) be a fixed sphere of radius a and a source of strength m be placed on z-axis at a distance f from the center of the sphere. R(r ≤ a), R′(r ≥ a) are two regions separated by the sphere S(r = a) (See figure 2.2).

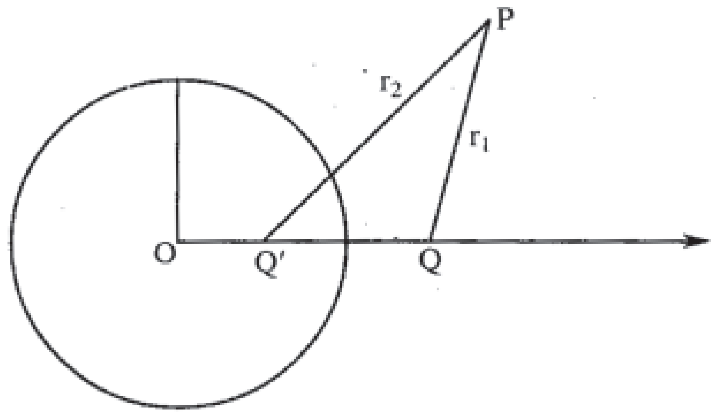

Figure 2.2

Let Q′ be the inverse point of Q with respect to the sphere then, $OQ' = \frac{a^2}{f}$.

Let P be any field point, which is at a distance r_1 and r_2 from Q and Q′ respectively and (r, θ, ω) be the co-ordinates of P.

The velocity potential ϕ is composed of two parts, one is ϕ_1 which is due to the source of strength m and another is ϕ_2 which is due to the presence of spherical boundary. The later part will be the velocity potential of the required system.

As the motion is irrotational, the velocity potential ϕ satisfies Laplace's equation

$$\nabla^2\phi = 0 \text{ in } R' \text{ except at } Q.$$

and the conditions

(i) $\phi \sim \frac{m}{r_1}$ near Q where r_1 is the distance from Q,

(ii) ϕ is regular at infinity

(iii) and since S is fixed, $\frac{\partial\phi}{\partial r} = 0$. on S.

Let us set

$$\phi = \frac{m}{r_1} + \phi_1$$

then

$$\nabla^2\phi_1 = 0 \text{ and } \frac{\partial\phi_1}{\partial r} = \frac{\partial\phi}{\partial r} - m\frac{\partial}{\partial r}\left(\frac{1}{r_1}\right) = -m\frac{\partial}{\partial r}\left(\frac{1}{r_1}\right) \text{ on S.}$$

Now,

$$\frac{1}{r_1} = \frac{1}{\sqrt{r^2 + f^2 - 2rf\cos\theta}} = \frac{1}{f}\frac{1}{\sqrt{\left(\frac{r}{f}\right)^2 - \frac{2f}{r}\cos\theta + 1}} = \frac{1}{f}\sum\frac{r^n}{f^n}P_n(\cos\theta)$$

where $P_n(\cos\theta)$ is Legendre's polynomial.

Again, $r > OQ' = \frac{a^2}{f} = b < a$

$$\frac{1}{r_2} = \frac{1}{\sqrt{r^2 + b^2 - 2rb\cos\theta}} = \frac{1}{f}\sum_{n=0}^{\infty}\frac{b^n}{r^{n+1}}P_n(\cos\theta).$$

On S,

$$\frac{\partial\phi_1}{\partial r} = -m\sum_{n=0}^{\infty}\frac{na^{n-1}}{f^{n+1}}P_n(\cos\theta).$$

Let us take

$$\phi_1 = \sum_{n=0}^{\infty}\frac{A_n}{r^{n+1}}P_n(\cos\theta)$$

so that is regular at infinity, and on S,

$$\left[\frac{\partial \phi_1}{\partial r}\right]_{r=a} = -\sum_{n=0}^{\infty} \frac{(n+1)A_n}{a^{n+2}} P_n(\cos\theta).$$

Thus, we obtain

$$-m\sum_{n=0}^{\infty} \frac{na^{n-1}}{f^{n+1}} P_n(\cos\theta) = -\sum_{n=0}^{\infty} \frac{(n+1)A_n}{a^{n+2}} P_n(\cos\theta).$$

Hence

$$A_n = \frac{mn}{n+1} \frac{a^{2n+1}}{f^{n+1}}.$$

Thus

$$\phi_1 = m\sum_{n=0}^{\infty} \frac{na^{2n+1}}{(n+1)f^{n+1}} \frac{P_n(\cos\theta)}{r^{n+1}} = m\sum_{n=0}^{\infty} \frac{a^{2n+1}}{f^{n+1}} \frac{P_n(\cos\theta)}{r^{n+1}} - m\sum_{n=0}^{\infty} \frac{a^{2n+1}}{(n+1)f^{n+1}} \frac{P_n(\cos\theta)}{r^{n+1}}$$

$$= \frac{m}{a}\sum_{n=0}^{\infty} \left(\frac{a^2}{f}\right)^{n+1} \frac{P_n(\cos\theta)}{r^{n+1}} - \frac{m}{a}\sum_{n=0}^{\infty} \frac{a^{2n+2}}{(n+1)f^{n+1}} \frac{P_n(\cos\theta)}{r^{n+1}}$$

$$\phi$$

$$= \frac{mb}{a}\sum_{n=0}^{\infty} \frac{b^n}{r^{n+1}} P_n(\cos\theta) - \frac{m}{a}\sum_{n=0}^{\infty} \frac{a^{2n+2}}{(n+1)f^{n+1}} \frac{P_n(\cos\theta)}{r^{n+1}} = \frac{ma}{fr_2} - \frac{m}{a}\sum_{n=0}^{\infty} \frac{b^{n+1}}{(n+1)r^{n+1}} P_n(\cos\theta)$$

$$= \frac{ma/f}{r_2} - \frac{m}{a}\int_0^b d\chi \sum_{n=0}^{\infty} \frac{\chi^n}{r^{n+1}} P_n(\cos\theta)$$

Set,

$$\lambda = \frac{1}{r'} = \frac{1}{(r^2 + \chi^2 - 2r\chi\cos\theta)^{1/2}} = \sum_{n=0}^{\infty} \frac{\chi^n}{r^{n+1}} P_n(\cos\theta).$$

Hence

$$\int_0^b \lambda d\chi = \int_0^b d\chi \sum_{n=0}^{\infty} \frac{\chi^n}{r^{n+1}} P_n(\cos\theta).$$

Therefore,

$$\phi_2 = \frac{ma/f}{r_2} - \frac{m}{a}\int_0^b \frac{d\chi}{r'}.$$