This shows that the required image consists of the source of strength $\frac{ma}{f}$ at the inverse point Q′ and a line distribution of sink of strength $-\frac{m}{a}$ per unit length extending from the center to the inverse point.

2.5.3 Image of a doublet in front of sphere :

Let a doublet of strength μ be placed at A on the z-axis, where OA = f and OA′ = f + δf so that mδf → μ, where m is the strength of source and sink. Let B and B′ be the inverse points of A and A′ respectively with respect to the sphere. The image of m at A is $\frac{ma}{f}$ at B and a line distribution of sink of strength $-\frac{m}{a}$ per unit length from O to B. The image of – m at A′ is $-\frac{ma}{f+\delta f}$ at B′, that is $-\frac{ma}{f}+\frac{ma\delta f}{f^2}$ and a line source of strength $\frac{m}{a}$ per unit length from 0 to B′.

Compounding this image system, we get a doublet of strength $\frac{ma}{f}BB'$, a source $ma\frac{\delta f}{f^2}$ and a sink $-\frac{m}{a}BB'$, all ultimately at the inverse point. Since $OB=\frac{a^2}{f}$, so $BB'=\frac{a^2\delta f}{f^2}$ so that the source and sink cancel each other and there remains a doublet of strength $\frac{ma}{f}\cdot\frac{a^2\delta f}{f^2}=\frac{ma^3\delta f}{f^3}$ i.e., $\frac{\mu a^3}{f^3}$ at the inverse point in the opposite direction to the given doublet.

2.6 Illustrative Solved Examples

Example 1 :

Show that when a sphere of radius a moves with uniform velocity U through a perfect incompressible infinite fluid, the acceleration of a particle of the fluid at (r, 0) is

$$3U^2\left(\frac{a^3}{r^4}-\frac{a^6}{r^7}\right).$$

Solution :

Superimpose a velocity – U both to the sphere and the liquid. This reduces the sphere to rest and the velocity potential of the flow is given by (Article 'Liquid steaming past a fixed sphere')

$$\phi = U\left(r + \frac{a^3}{2r^2}\right)\cos\theta. \tag{1}$$

Also

$$\dot{r} = -\frac{\partial\phi}{\partial r} = -U\left(1 - \frac{a^3}{r^3}\right)\cos\theta \tag{2}$$

and

$$r\dot{\theta} = -\frac{1}{r}\frac{\partial\phi}{\partial\theta} = U\left(1 + \frac{a^3}{r^3}\right)\sin\theta. \tag{3}$$

Again, from (2), we have

$$\ddot{r} = U\left(1 - \frac{a^3}{r^3}\right)\sin\theta\dot{\theta} - U\frac{3a^3}{r^4}\dot{r}\cos\theta$$

$$= U\left(1 - \frac{a^3}{r^3}\right)\sin\theta\dot{\theta} + \frac{3a^4}{r^4}U^2\left(1 - \frac{a^3}{r^3}\right)\cos^2\theta, \text{ by (2).}$$

Clearly for a point (r, 0), the velocity is only along the direction of r and hence the acceleration will also be only along r.

Thus the required acceleration

$= \ddot{r}$ only at (r, 0)

$= \frac{3a^2}{r^4}U^2\left(1 - \frac{a^3}{r^3}\right)$, from (3) with $\theta = \dot{\theta} = 0$

$= 3U^2\left(\frac{a^3}{r^4} - \frac{a^6}{r^7}\right).$

Example 2 :

A stream of water of greater depth is flowing with a uniform velocity U over a plane level bottom. A hemisphere of weight W in water and radius a, rests with its base on the bottom. Prove that the average pressure between the base of the hemisphere and the bottom is less than the fluid pressure at any point of the bottom at a great distance from the hemisphere if

$$U^2 = \frac{32W}{11\pi a^2 \rho}.$$

Solution :

Let water be flowing past a fixed hemisphere with velocity U along z-axis and (r, θ, ω) be the spherical polar co-ordinates of a point referred to the center of the hemisphere as the origin.

The velocity potential is given by

$$\phi = U\left(r + \frac{a^3}{2r^2}\right)\cos\theta. \tag{1}$$

Then

$$\left(\frac{\partial\phi}{\partial r}\right)_{r=a} = \left[U\left(1 - \frac{a^3}{r^3}\right)\cos\theta\right]_{r=a} = 0.$$

$$\left(\frac{1}{r}\frac{\partial\phi}{\partial r}\right)_{r=a} = \left[-U\left(1 + \frac{a^3}{r^3}\right)\sin\theta\right]_{r=a} = -\frac{3}{2}U\sin\theta.$$

Let q be the velocity at any point of the boundary of the sphere r = a. Then, we have

$$q^2 = \left\{\left(-\frac{\partial\phi}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial\phi}{\partial r}\right)\right\}_{r=a} = \frac{9}{4}U^2 \sin^2\theta. \tag{2}$$

In steady motion in absence of external forces, the pressure at any point by Bernoulli's equation is given by

$$\frac{p}{\rho} + \frac{1}{2}q^2 = C. \tag{3}$$

But p = Π, q = U at infinity. So (3) gives

$$\frac{\Pi}{\rho} + \frac{1}{2}U^2 = C. \tag{4}$$

Subtracting (4) from (3), we obtain

$$p = \Pi + \frac{1}{2}\rho U^2 - \frac{1}{2}\rho q^2. \tag{5}$$

Using (2), the pressure p′ at any point P on the surface of the sphere r = a is given by

$$p' = \Pi + \frac{1}{2}\rho U^2 - \frac{9}{8}\rho U^2 \sin^2\theta. \tag{6}$$

Relation between (x, y, z) and (r, θ, ω) are given by

$$x = r\sin\theta\cos\omega,\ \ y = r\sin\theta\sin\omega;\ \ z = r\cos\theta. \tag{7}$$

Direction cosine of OP are (x/r, y/r, z/r) where OP = r = a. Using (2), direction cosine of OP are (sin θ cos θ, sin θ sin θ, cos θ).

Hence the component of p′ along x-axis is p′ sin θ cos ω.

Taking a sin θdω, adθ as an element on the surface of the hemisphere, the total thrust on the hemisphere due to water along OX

$$= \int_{\theta=0}^{\pi}\int_{\omega=-\pi/2}^{\pi/2}(p'\sin\theta\cos\omega)(a\sin\theta d\omega . a d\theta)$$

$$= a^2\int_0^{\pi}\int_{-\pi/2}^{\pi/2}\left[\Pi + \frac{1}{2}\rho U^2 - \frac{9}{8}\rho U^2\sin^2\theta\right]\sin^2\theta\cos\omega d\theta d\omega \ \ [\text{using (1)}]$$

$$= 2a^2\int_0^{\pi}\left[\Pi + \frac{1}{2}\rho U^2 - \frac{9}{8}\rho U^2\sin^2\theta\right]\sin^2\theta d\theta$$

$$= 2a^2\int_0^{\pi}\left[\left(\Pi + \frac{1}{2}\rho U^2\right) - \frac{9}{8}\rho U^2\frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma(3)}\right]$$

$$= \pi a^2\left(\Pi - \frac{11\rho U^2}{32}\right).$$

Since there is a weight W on the base, the total thrust on the base

$$= \pi a^2\left(\Pi - \frac{11\rho U^2}{32}\right) + W.$$

Therefore,

$$\text{average pressure on the base} = \frac{\text{pressure on base}}{\text{area of the base}} = \Pi - \frac{11\rho U^2}{32} + \frac{W}{\pi a^2}.$$

Hence,

the average pressure < pressure at great distance

if

$$\Pi - \frac{11\rho U^2}{32} + \frac{W}{\pi a^2} < \Pi$$

i.e., if

$$U^2 > \frac{32W}{11\rho\pi a^2}.$$

Example 3 :

Incompressible fluid of density ρ, is contained between two rigid concentric spherical surfaces, the outer one of mass M_1 and radius a, the inner one of mass M_2 and radius b. A normal blow P is given to the outer surface. Prove that the initial velocities of the two containing surface (U for the outer and V for the inner) are given by the equations

$$\left\{M_1 + \frac{2\pi\rho a^3(2a^3+b^3)}{3(a^3-b^3)}\right\}U - \frac{2\pi\rho a^3 b^3}{a^3-b^3}V = P$$

$$\left\{M_2 + \frac{2\pi\rho b^3(2b^3+a^3)}{3(a^3-b^3)}\right\}V = \frac{2\pi\rho a^3 b^3}{a^3-b^3}U.$$

Solution :

As in article 'Moving Concentric Sphere', we have

$$\phi = \frac{1}{a^3-b^3}\left[(Vb^3 - Ua^3)r + \frac{(V-U)a^3b^3}{2r^2}\right]\cos\theta \qquad (1)$$

The normal blow P in the outer imparts velocity U to the outer and V to the inner spherical surface. Let ϖ_1, ϖ_2 be the impulsive pressure on an element ds of the boundary surface r = a and r = b respectively. Then

$$M_1U = P - \iint \varpi_1 \cos\theta\, ds \text{ on } r = a \qquad (2)$$

and

$$M_2 U = P - \iint \varpi_2 \cos\theta \, ds \text{ on } r = b \tag{3}$$

On r = a, from (1)

$$\varpi_1 = (\rho\phi)_{r=a} = \frac{1}{a^3 - b^3}\left[(Vb^3 - Ua^3)a + \frac{(V-U)ab^3}{2}\right]\cos\theta.$$

Hence (2) reduce to

$$M_1 U = P - \int_0^\pi \varpi_1 \cos\theta . ad\theta . 2\pi a \sin\theta$$

$$= P - \frac{2\pi a^3}{a^3 - b^3}\left[(Vb^3 - Ua^3)a + \frac{(V-U)ab^3}{2}\right] \times \int_0^\pi \cos^2\theta \sin\theta d\theta$$

$$= P - \frac{\pi a^3}{a^3 - b^3}\left[3Vb^3 - U(2a^3 + b^3)\right] \times \left(-\frac{2}{3}\right).$$

Therefore,

$$\left\{M_1 + \frac{2\pi\rho a^3 (2a^3 + b^3)}{3(a^3 - b^3)}\right\}U - \frac{2\pi\rho a^3 b^3}{a^3 - b^3}V = P. \tag{4}$$

Again, on r = b

$$\varpi_2 = (\rho\phi)_{r=b} = \frac{1}{a^3 - b^3}\left[(Vb^3 - Ua^3)b + \frac{(V-U)a^3 b}{2}\right]\cos\theta.$$

Hence (3) reduces to

$$M_2 V = -\int_0^\pi \varpi_2 \cos\theta . ad\theta . 2\pi b \sin\theta$$

$$-\frac{2\pi b^2}{a^3 - b^3}\left[(Vb^3 - Ua^3)b + \frac{(V-U)a^3 b}{2}\right] \times \int_0^\pi \cos^2\theta \sin\theta d\theta$$

$$= -\frac{2\pi b^3 \rho}{a^3 - b^3}\left[Vb^3 - Ua^3 + \frac{1}{2}a^3(U - V)\right] \cdot \left(-\frac{2}{3}\right).$$

Therefore,

$$\left\{M_2 + \frac{2\pi\rho b^3(2b^3 + a^3)}{3(a^3 - b^3)}\right\}V = \frac{2\pi\rho a^3 b^3}{a^3 - b^3}U.$$

Example 4 :

Prove that if two rigid surface of revolution one of which surrounds the other, are moving along their common axis with velocities U_1, U_2 and space between them filled with homogenous liquid, the momentum of the liquid is $M_2U_2 - M_1U_1$, where M_1, M_2 are the masses of liquid which either surface would contain.

Solution :

Let x-axis be taken as the axis of revolution. Due to symmetry, the moment of momentum of the liquid along the y-axis and z-axis is zero. The momentum of the liquid along x-axis is

$$\iiint \rho u\,dxdydz. \tag{1}$$

If ϕ be the potential at any point P(x, y, z) of the liquid, then $u = -\frac{\partial\phi}{\partial x}$ and so (1) becomes

$$-\iiint \rho\frac{\partial\phi}{\partial x}dxdydz. \tag{2}$$

the integration extends over the whole volume of the liquid.

Using the Green's theorem (2) can be re-written as

$$\iint x\frac{\partial\phi}{\partial x}ds. \tag{3}$$

where δn is an element of the outward normal at the element of the bounding surface δs.

Hence the momentum of the liquid along x-axis is

$$= \rho\iint_{inner} x\frac{\partial\phi}{\partial x}ds_1 + \rho\iint_{outer} x\frac{\partial\phi}{\partial x}ds_2$$

$$= -\rho\iint_{inner} xl_1U_1ds_1 + \rho\iint_{outer} xl_2U_2ds_2 \tag{4}$$

where l_1 and l_2 are cosines of the angles which the outer drawn normals at ds_1, ds_2 make with x-axis.

But $l_1 ds_1 = dxdy$ and $l_2 ds_2 = dydz$. So (4) reduces to :

The momentum of the liquid along x-axis

$$= -\rho \int\int xU_1 dxdy + \rho \int\int xU_2 dydz$$

$$= -U_1 \int\int \rho x dxdy + U_2 \int\int \rho x dydz$$

$$= M_2 U_2 - M_1 U_1,$$

where M_1, M_2 are the masses of the liquids which either surface would contain.

2.7 Model Questions

Short Questions :

1. Find the solution of Laplace's equation in spherical polar coordinates having axial symmetry.
2. Define Stokes' stream function.
3. Define source, sink and doublet. Hence find the velocity potential for each of them.

Broad Questions :

1. Introducing Stokes' stream function, discuss the irrotational axi-symmetric motion of an ideal liquid.
2. A solid moves along the axis of revolution OX with velocity U in a non-viscous liquid the motion of the liquid being symmetrical about OX and irrotational. Discuss the motion.
3. Find the expression for the velocity potential and the equation of stream lines for the irrotational motion of a non-viscous liquid at rest at infinity in which a sphere is moving with uniform velocity, the motion being symmetrical about z-axis.
4. Deduce the equation of motion of a sphere moving in an incompressible ideal fluid at rest at infinity with velocity U along the axis of z. Hence show that the effect of the presence of the liquid is to reduce the external force in the ratio $(\sigma - \rho) : (\sigma + \frac{1}{2}\rho)$, σ and ρ being the densities of the sphere and the liquid respectively.

5. Discuss the irrotation motion of an ideal liquid past a fixed sphere in a uniform stream. Hence find the equation of the lines of flow.

6. The region between two concentric spheres is filled with a homogeneous incompressible fluid, the surfaces of the spheres being subjected to given impulses in the z-direction so that the two spheres start to move with given velocities in the positive direction at the z-axis. Determine the resulting motion.

7. Find the image of a source (or sink or doublet) with respect to a rigid plane.

8. Find the image of a source (or sink or doublet) in front of a sphere.

Problems :

1. An infinite ocean of an incompressible perfect liquid of density ρ is streaming past a fixed spherical obstacle of radius a. The velocity is uniform and equal to U except in so far as it is disturbed by sphere, and the pressure in the liquid at a great distance from the obstacles is Π. Show that the thrust on that half of the sphere on which the liquid impinges is

$$\pi a^2 \cdot \left\{ \Pi - \frac{\rho U^2}{16} \right\}.$$

2. Find the pressure at any point of a liquid, of infinite extent and at rest a great distance, through which a sphere is moving under no external forces with constant velocity U, and show that the mean pressure over the sphere is in defect of the pressure Π at a great distance by $\frac{1}{4}\rho U^2$, it being supposed that Π is sufficiently large for the pressure everywhere to be positive, that is, that

$$\Pi > \frac{5}{8}\rho U^2.$$

3. Liquid of density ρ fills the space between a solid sphere of radius a and density ρ′ and a fixed concentric spherical envelope of radius b; prove that the work done by an impulse which starts the solid sphere with velocity V is

$$\frac{1}{3}\pi a^3 V^3 \left(2\rho' + \frac{2a^3 + b^3}{b^3 - a^3}\rho \right).$$

4. The space between two concentric spherical shells of radii a and b$(a > b)$ is filled with an incompressible fluid of density ρ and the shells suddenly begin to move with velocities U, V in the same direction : prove that the resultant impulsive pressure on the inner shell is

$$\frac{2\pi\rho b^2}{3(a^3 - b^3)}\{3a^3U - (a^3 + 2b^3)V\}.$$

5. A sphere of radius a is made to move in incompressible perfect fluid with non-uniform velocity u along x-axis. If the pressure at infinity is zero, prove that at a point x in advance of the center

$$p = \frac{1}{2}\rho a^3\left\{\frac{\dot{u}}{x^2} + u^2\left(\frac{2}{x^3} - \frac{a^3}{x^6}\right)\right\}.$$

2.8 Summary

In this chapter, we have considered the three-dimensional irrotational motion of an ideal liquid with special reference to a sphere and a solid of revolution. Notion of source, sink, doublet and their images with respect to a rigid plane and a sphere has also been introduced.

Unit 3 □ Vortex Motion

Structure

3.0 Introduction

3.1 Vortex lines and Vortex tubes

3.2 Rectilinear Vortex

3.3 Circular Vortex

3.3.1 Vortex pair

3.3.2 Vortex doublet

3.4 Infinite row of parallel rectilinear vortices

3.4.1 Single infinite row

3.4.2 Infinite row of parallel rectilinear vortices (Karman Vortex Street)

3.5 Examples

3.6 Model Questions

3.0 Introduction

It is well known that for irrotational motion the velocity vector $\mathbf{q} = (u,v,w)$ can be represented in the form of the gradient of a velocity potential ϕ as

$$\mathbf{q} = \text{grad}\ \phi$$

or, in other words,

$$u = \frac{\partial\phi}{\partial x},\ v = \frac{\partial\phi}{\partial y},\ w = \frac{\partial\phi}{\partial z}. \tag{1}$$

The *vorticity* is defined to be a vector $\Omega = \text{curl}\ \mathbf{q}$, whose components are

$$\Omega_x = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z},\ \Omega_y = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x},\ \Omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}. \tag{2}$$

The above components vanish when the conditions (1) are satisfied. Thus, for an irrotational motion when $\mathbf{q} = \text{grad}\phi$,

$$\Omega = \text{curl grad}\ \phi = 0. \tag{3}$$

Conversely, if $\Omega = 0$, then with the aid of vector analysis, it can be shown that equation (1) will always hold. Thus, in irrotational motion, a velocity potential certainly exists.

This chapter will consist of investigation of such motions of a fluid for which the *vorticity vector* Ω is different from zero at least in some part of the fluid under consideration. We will call such motions as vortex motions of the fluid.

3.1 Vortex lines and Vortex tubes

A vortex line is a curve in the fluid such that its tangent at any point gives the direction of the local vorticity. Therefore, the equations of a vortex line have the form

$$\frac{dx}{\Omega_x} = \frac{dy}{\Omega_y} = \frac{dz}{\Omega_z} \tag{4}$$

where Ω_x, Ω_y, Ω_z are the components of the vorticity vector Ω. Note that, the above equations are analogous to the equations for a streamlines. Portions of the fluid bounded by vortex lines through every point of an infinitely small closed curves are called vortex filaments, or simply vortices. Vortex lines passing through any closed curve form a tubular surface, which is called a *vortex tube*. The fluid contained within such a tube constitutes what is called a vortex-filament. Let δS_1, δS_2 be two sections of a vortex tube and let $\mathbf{n_1}$ and $\mathbf{n_2}$ be the unit normals to these sections drawn outwards from the fluid between them. Also, let δS be the curved surface of the vortex tube. Then, $\Delta S = \delta S_1 + \delta S_2 + \delta S$ = total surface area of the element. Let ΔV be the total volume contained in ΔS. Then

$$\int_{\Delta S} \mathbf{n}.\Omega dS = \int_{\Delta V} \operatorname{div}\Omega dV = 0,$$

since div $\Omega = 0$. Thus

$$\int_{\delta S_1} \mathbf{n}.\Omega dS = \int_{\delta S} \mathbf{n}.\Omega dS + \int_{\delta S_2} \mathbf{n}.\Omega dS = 0.$$

At each point of δS, $\mathbf{n}.\Omega = 0$, since Ω is tangential to the curved surface. Thus

$$(\mathbf{n_1}.\Omega)\delta S_1 + (\mathbf{n_2}.\Omega)\delta S_2 = 0$$

approximately to the first order (using the mean value theorem of integral calculus). This shows that $|\mathbf{n}.\Omega| dS$ is constant for every section δS of the vortex tube. Its value is called the *strength* of the vortex tube. A vortex tube whose strength is unity is called a *unit vortex tube*.

Some properties of vertices :

(1) Vortex lines and tubes move with the fluid.

Let C be any closed curve drawn on the surface of the vortex tube containing an area S of the tube and not embracing the tube. As the vortieity vectors are everywhere lying on the surface S, it follows that, $\mathbf{n}.\Omega = 0$. So the circulation Γ around C is given by

$$\int_{\Gamma} \mathbf{q}.ds = \int_{S} \mathbf{n}.\Omega dS = 0.$$

After an interval of time, the same fluid particles form a new surface, say S'. According to Kelvin's theorem, the circulation around S' must also be zero. As this is true for any S, the component of vortieity normal to every element of S' must vanish, showing that S' must lie on the surface of the vortex tube. Hence, vortex lines and vortex tubes move with fluid.

(2) Vortex lines and tubes move with the fluid.

Let C be any closed curve drawn on the surface of the vortex tube containing an area S of the tube and not embracing the tube. As the vorticity vectors are everywhere lying on the surface S, it follows that $\mathbf{n}.\Omega = 0$. So the circulation Γ around C is given by

$$\int_{\Gamma} \mathbf{q}.ds = \int_{S} \mathbf{n}.\Omega dS = 0.$$

After an interval of time, the same fluid particles form a new surface, say S'. According to Kelvin's theorem, the circulation around S' must also be zero. As this is true for any S, the component of vorticity normal to every element of S' must vanish, showing that S' must lie on the surface of the vortex tube. Hence, vortex lines and vortex tubes move with fluid.

(3) A vortex tube cannot originate or end within the fluid. It must either end at a solid boundary or form a closed loop (a 'vortex ring').

Suppose S is any closed surface containing a volume V. Then

$$\int_{S} \mathbf{n}.\Omega dS = \int_{V} \operatorname{div} \Omega dV = 0. \tag{5}$$

Equation (5) shows that the total strength of vortex tubes emerging from S is equal to that entering S. This means that *vortex lines and tubes cannot originate* or *terminate at internal points in a fluid.* They can only form closed curves or terminate on boundaries.

(4) Strength of a vortex tube remains constant for all time.

If C is a closed curve embracing once the vortex tube and if S denotes an area contained in C, then the circulation Γ of the fluid velocity **q** around the vortex tube is defined as

$$\Gamma = \oint_C \mathbf{q}.ds \tag{6}$$

Then, by Stokes' theorem

$$\Gamma = \int_S \mathbf{n}.qdS. \tag{7}$$

Equation (7) shows that Γ is nothing but the strength of vortex tube with surface area S. Since for an inviseid fluid the circulation around any closed curve in the fluid moving along with the fluid, remains constant in time, therefore strength of the vortex also remains constant in time.

The above theorems are known as **Helmholtz's vortex theorems :**

We shall assume that the fluid is a single-valued function of time only.

3.2 Rectilinear Vortex

Consider a single tube whose cross-section is a circle of radius a and with its axis parallel to the axis of z surrounded by unbounded fluid. The motion is similar in all planes parallel to xy and it has no velocity along the axis of z. By making the area contained within the tube sufficiently small we see that the distribution producing such a flow must be uniform along the z-axis. Such a distribution along the z-axis is called a uniform *rectilinear* or *line vortex.* Thus if **q** = (u, v, w) be the velocity, then w = 0 and u, v are independent of z. If $\Omega = (\Omega_x, \Omega_y, \Omega_z)$ be the vorticity vector, then

$$\Omega_x = 0, \Omega_y = 0, \Omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}. \tag{8}$$

The velocity components u, v are related to the stream function ψ by

$$u = -\frac{\partial \psi}{\partial y} \text{ and } v = \frac{\partial \psi}{\partial x}. \tag{9}$$

Use of (9) in (8) gives

$$\Omega_z = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}. \tag{10}$$

Thus, ψ satisfies

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \begin{cases} \Omega_x, & \text{on the vortex,} \\ 0, & \text{out side the vortex.} \end{cases} \tag{11}$$

Let P(r, θ) be any point outside the vortex. Since the motion outside the vortex is irrotational, the velocity potential ϕ exists and

$$\frac{\partial \psi}{\partial r} = \frac{1}{r}\frac{\partial \phi}{\partial \theta} \tag{12}$$

holds, r, θ being polar coordinates. Since, in the region out side the vortex ψ is harmonic so we get

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2} = 0. \tag{13}$$

If the motion is symmetric about the origin, ψ must be independent of θ. Then equation (13) reduces to

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\psi}{dr}\right) = 0$$

giving

$$\psi = c \log r, \ c = \text{constant.} \tag{14}$$

Using the relation of ϕ and ψ given by (12) we get

$$\phi = -c\theta. \tag{15}$$

Thus the complex potential function w is given by

$$w = \phi + i\psi = -c\theta + ic \log r = ic \log z. \tag{16}$$

Let k be the circulation in the circuit enclosing the vortex. Then

$$k = \int_0^{2\pi}\left(-\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right) r d\theta = 2\pi c$$

so that

$$c = \frac{k}{2\pi}$$

and hence w is given by

$$w = \frac{ik}{2\pi} \log z. \tag{17}$$

This is the complex potential due to a vortex of strength k placed at the origin. If the vortex be placed at $z_0 = x_0 + iy_0$, instead of (0, 0), then the complex potential w has the form

$$w = \frac{ik}{2\pi} \log(z - z_0). \tag{18}$$

Let $P(z) \equiv P(x, y)$ be another point in the fluid other than (x_0, y_0). Then distance r_0 between (x, y) and (x_0, y_0) is given by

$$r_0^2 = (x - x_0)^2 + (y - y_0)^2. \tag{19}$$

From (18), we see that the stream function ψ is given by

$$\psi = \frac{k}{2\pi} \log r_0.$$

Thus,

$$u = -\frac{\partial \psi}{\partial y} = -\frac{\partial \psi}{\partial r_0}\frac{\partial r_0}{\partial y} = -\frac{k}{2\pi} \cdot \frac{y - y_0}{r_0^2}$$

and

$$v = \frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial r_0}\frac{\partial r_0}{\partial x} = \frac{k}{2\pi} \cdot \frac{x - x_0}{r_0^2}.$$

Thus the magnitude of the velocity **q** is given by

$$q = \left(u^2 + v^2\right)^{\frac{1}{2}} = \frac{k}{2\pi r_0}.$$

This is the velocity at any point P(x, y) due to presence of a vortex of strength k at (x_0, y_0).

Note :

If there be any number of vertices of strength k_s at z_s, s = 1, 2, 3, ..., then the complex potential at any point z in the fluid is given by

$$w = \frac{i}{2\pi}\sum_s k_s \log(z - z_s),$$

and the velocity components are given by

$$u = -\frac{1}{2\pi}\sum_s k_s \frac{(y - y_s)}{r_s^2} \text{ and } v = \frac{1}{2\pi}\sum_s k_s \frac{(x - x_s)}{r_s^2}$$

where

$$z_s = x_s + iy_s \text{ and } r_s^2 = (x - x_s)^2 + (y - y_s)^2.$$

Let (u_s, v_s) denote the velocity components of the vortex of strength k_s. Then

$$u_s = -\frac{1}{2\pi}\sum_{r\neq s} k_r \frac{(y_r - y_s)}{R_{r_s}^2} \text{ and } v_s = \frac{1}{2\pi}\sum_{r\neq s} k_r \frac{(x_r - x_s)}{R_{r_s}^2}$$

where

$$R_{r_s}^2 = (x_r - x_s)^2 + (y_r - y_s)^2.$$

Note that the expressions $\sum k_s u_s$ and $\sum k_s v_s$ will consist of pairs of terms of the forms

$$k_r \cdot \frac{k_s}{2\pi}\frac{(x_r - x_s)}{R_{r_s}^2} \text{ and } k_s \cdot \frac{k_r}{2\pi}\frac{(x_s - x_r)}{R_{r_s}^2}$$

and as such

$$\sum k_s u_s = 0 \text{ and } \sum k_s v_s = 0.$$

Hence, regarding k as a mass, the center of gravity of the vortex system, viz.

$$\bar{x} = \frac{\sum k_s u_s}{\sum k_s}, \quad \bar{y} = \frac{\sum k_s v_s}{\sum k_s}$$

remains stationary throughout the motion. Note that if $\sum k_s = 0$, the center $(\bar{x}, \bar{y})$ is at infinity.

3.3 Circular Vortex

Let there be a single cylindrical vortex tube, whose cross-section is a circle of radius a, surrounded by unbounded fluid.

The section of the vortex by the plane of the motion is a circle and the arrangement may therefore be referred to as a *circular vortex*.

3.3.1 Vortex pair

Consider the case of two vortices of strengths k_1 and k_2 at a distance r_0 apart. Let A, B be their centers, O, the center of the system. The point O divides AB in the ratio $k_2 : k_1$. The motion of each vortex as a whole is entirely due to the other, and is therefore always perpendicular to AB. Hence the two vortices remain ahvays at the same distance from one another and rotate with constant angular velocity about O which is fixed. The velocities at the two vortices at A and B are respectively $\frac{k_1}{2\pi r_0}$ and $\frac{k_2}{2\pi r_0}$. To obtain the angular velocity w of the system, we divide the velocity of the vortex A by the distance AO, where

$$AO = \frac{k_2}{k_1 + k_2} . AB = \frac{k_2 r_0}{k_1 + k_2} .$$

Therefore, the angular velocity is given by

$$\omega = \frac{\text{velocity of the vortex at A}}{AO} = \frac{k_1 + k_2}{2\pi r_0^2} .$$

If k_1, k_2 be of the same sign, i.e. if the direction of rotation in the two vortices be the same then O lies between A and B; otherwise O lies in AB or BA, produced. If $k_1 = -k_2$, O is at infinity. However, A, B move with equal velocities $\frac{k_1}{2\pi r_0}$ at right angles to AB, which remains fixed in direction. Such a combination of two equal and opposite vortices may be called a *vortex pair*.

3.3.2 Vortex doublet

Consider a vorlex pair, k at $ae^{i\alpha}$ and $-$ k at $-ae^{i\alpha}$ in the complex z-plane where $z = x + iy$. If we let $a \to 0$ and $k \to \infty$ so that $2ak = \mu$ is a finite constant, we get a vortex doublet of strength μ inclined at an angle α to the x-axis.

The direction of the doublet is determined from the vortex of negative rotation to that of positive rotation. The complex potential is

$$w = \lim_{a\to 0} \frac{ik}{2\pi}\{\log(z - ae^{i\alpha}) - \log(z + ae^{i\alpha})\}$$

$$= \lim_{a\to 0} \frac{ik}{2\pi}\left(-\frac{ae^{i\alpha}}{z} + \frac{a^2e^{2i\alpha}}{2z^2} - \cdots - \frac{ae^{i\alpha}}{z} - \frac{a^2e^{2i\alpha}}{2z^2} - \cdots\right) = -\frac{i\mu}{2\pi z}e^{2i\alpha}.$$

The stream function is $\psi = -\frac{\mu}{2\pi r}\cos(\alpha - \theta)$.

If, in particular, we take the vortex doublet to be at the origin and along the axis of y, we have $\psi = -\frac{\mu \sin\theta}{2\pi r}$. If we put $\frac{\mu}{2\pi} = Ub^2$, we obtain $\psi = -\frac{Ub^2 \sin\theta}{r}$ which is the stream function for a circular cylinder of radius b moving with velocity U along the x-axis.

Thus the motion due to a circular cylinder is the snme as that due to a suitable vortex doublet placed at the center, and with its axis perpendicular to the direction of motion.

3.4 Infinite row of parallel rectilinear vortices

3.4.1 Single infinite row

Consider an infinite row of vortices each of strength k at the points 0, ±a, ±2a, ..., ±na, ... (as shown in figure 3.1).

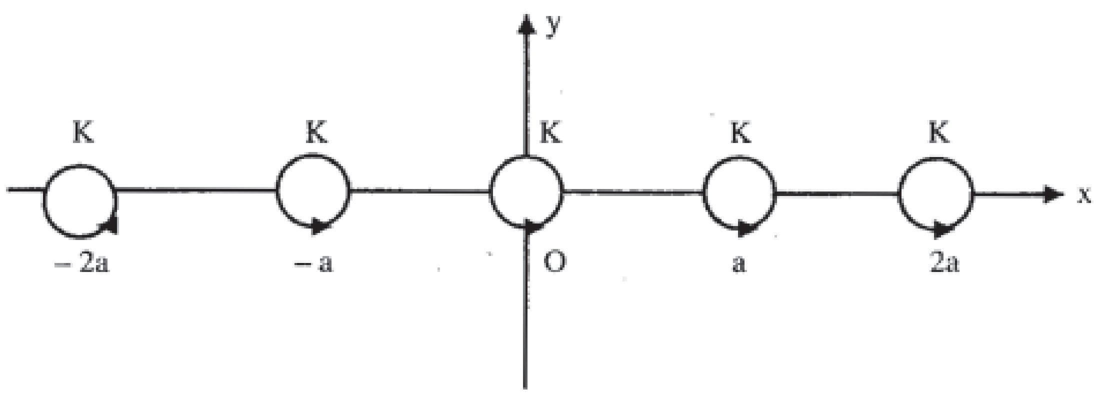

Figure 3.1

The complex potential of the (2n + 1) vortices nearest to the origin is

$$w_n = \frac{ik}{2\pi}\log z + \frac{ik}{2\pi}\log(z-a) + \cdots + \frac{ik}{2\pi}\log(z-na)$$

$$+\frac{ik}{2\pi}\log(z+a) + \cdots + \frac{ik}{2\pi}\log(z+na)$$

$$= \frac{ik}{2\pi}\log\{z(z^2-a^2)(z^2-2^2a^2)\cdots(z^2-n^2a^2)\}$$

$$= \frac{ik}{2\pi}\log\left\{\frac{\pi z}{a}\left(1-\frac{z^2}{a^2}\right)\left(1-\frac{z^2}{2^2a^2}\right)\cdots\left(1-\frac{z^2}{n^2a^2}\right)\right\} + \frac{ik}{2\pi}\log\left\{\frac{a}{\pi}.a^2.2^2a^2\ldots n^2a^2\right\}.$$

The constant term may be omitted, so that we write

$$w_n = \frac{ik}{2\pi}\log\left\{\frac{\pi z}{a}\left(1-\frac{z^2}{a^2}\right)\left(1-\frac{z^2}{2^2a^2}\right)\cdots\left(1-\frac{z^2}{n^2a^2}\right)\right\}. \qquad (20)$$

Now, sin x can be expressed as an infinite product in the form

$$\sin x = x\left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{2^2\pi^2}\right)\cdots\left(1-\frac{x^2}{n^2\pi^2}\right)\cdots \qquad (21)$$

Thus letting $n \to \infty$ in (20), we get by virtue of (21),

$$w = \frac{ik}{2\pi}\log\sin\left(\frac{\pi z}{a}\right). \qquad (22)$$

Consider the vortex at $z = 0$. Since its moton is due to the other vortices, the complex velocity of the vortex at the origin is given by

$$-\frac{d}{dz}\left\{\frac{ik}{2\pi}\log\sin\frac{\pi z}{a} - \frac{ik}{2\pi}\log z\right\}_{z=0} = -\frac{ik}{2\pi}\left(\frac{\pi}{a}\cdot\cot\frac{\pi z}{a} - \frac{1}{z}\right)_{z=0} = 0.$$

Thus the vortex at the origin is at rest. Similarly it can be shown that the remaining vortices are also at rest. Thus the vortex row induces no velocity in itself.

To determine the stream function we note that

$$w(z) = \phi + i\psi,\ \overline{w}(\overline{z}) = \phi - i\psi$$

so that from (22)

$$2i\psi = w(z) - \overline{w}(\overline{z}) = \frac{ik}{2\pi}\log\left\{\sin\frac{\pi z}{a}\sin\frac{\pi\overline{z}}{a}\right\},$$

$$\psi = \frac{k}{4\pi}\log\frac{1}{2}\left(\cosh\frac{2\pi y}{a} - \cos\frac{2\pi x}{a}\right).$$

For large values of $\frac{y}{a}$, we neglect the term $\cos\frac{2\pi x}{a}$, for its modulus never exceeds unity, and therefore along the streamlines ψ = constant. Thus at a great distance from the row the stream lines are parallel to the row.

Again, if v_1, v_2 are the complex velocities at the points z, $\bar{z}$ respectively, we have

$$v_1 + v_2 = -\frac{d}{dz}\left\{\frac{ik}{2\pi}\log\sin\frac{\pi z}{a}\right\}_{z=z} - \frac{d}{dz}\left\{\frac{ik}{2\pi}\log\sin\frac{\pi z}{a}\right\}_{z=\bar{z}}$$

$$= -\frac{ik}{2a}\cot\frac{\pi z}{a} - \frac{ik}{2a}\cot\frac{\pi \bar{z}}{a} = -\frac{ik}{2a}\frac{2\sin\frac{2\pi x}{a}}{\cosh\frac{2\pi y}{a} - \cos\frac{2\pi x}{a}},$$

which is purely imaginary and tends to zero when y tends to infinity. Thus the velocities along the distant streamlines are parallel to the row but in opposite directions.

3.4.2 Infinite row of parallel rectilinear vortices (Karman Vortex Street)

This consists of two parallel infinite rows of the same spacing, say a, but of opposite vortex strengths k and – k, so arranged that each vortex of the upper row is directly above

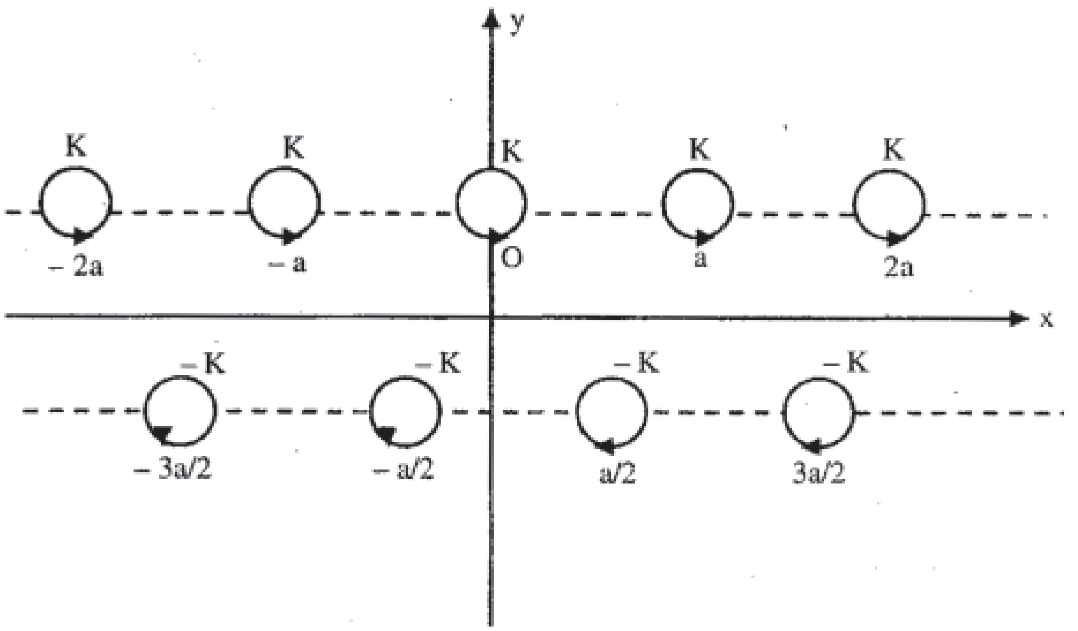

Figure 3.2

the mid point of the line joining two vortices of the lower row and vice-versa. Taking the configuration at time t = 0, we take the axes as shown in the figure 3.2, the x-axis being midway between and parallel to the rows which are at the distance b apart. At this instant the vortices in the upper row are at the points $ma + \frac{1}{2}ib$, and those in the lower row at the points $\left(m+\frac{1}{2}\right)a - \frac{1}{2}ib$, where $m = 0, \pm 1, \pm 2, \ldots$

The complex potential at the instant t = 0, by the preceding section is given by

$$w = \frac{ik}{2\pi}\log\sin\frac{\pi}{a}\left(z - \frac{ib}{2}\right) + \frac{ik}{2\pi}\log\sin\frac{\pi}{a}\left(z - \frac{a}{2} + \frac{ib}{2}\right).$$

Since neither row induces any velocity in itself, the velocity of vortex at $z = \frac{a}{2} - \frac{ib}{2}$ will be given by

$$-u_1 + iv_1 = \left[\frac{d}{dz}\frac{ik}{2\pi}\sin\frac{\pi}{a}\left(z - \frac{ib}{2}\right)\right]_{z=\frac{a}{2}-\frac{ib}{2}}$$

$$= \frac{ik}{2a}\cot\left(\frac{\pi}{2} - \frac{i\pi b}{a}\right) = -\frac{k}{2a}\tanh\frac{\pi b}{a}.$$

Thus the lower row advances with velocity

$$V = \frac{k}{2a}\tanh\frac{\pi b}{a},$$

and similarly the upper row advances with the same velocity. The rows will advance the distance a in time $\tau = \frac{a}{V}$ and the configuration will be the same after this interval as at the initial instant.

Note :

In a Karman vortex street, under the influence of some operation, all or certain of the vortices may experience small displacements. Then it is possible that with the passage of time the vortices will remain close to the positions which they would have had if they had not been subject to displacements. We then say that the motion is stable. If, however, the displaced vortices tend to move away from the position corresponding to unperturbed state, the motion will be called unstable. A necessary condition of stability for the Karman's vortex street is

$$\cosh \frac{b\pi}{a} = \sqrt{2}$$

so that $b = 0.281a$.

3.5 Illustrative Solved Examples

Example 1

If

$$u = \frac{ax - by}{x^2 + y^2}, \; v = \frac{ay + bx}{x^2 + y^2}, \; w = 0,$$

investigate the nature of motion of the liquid.

Solution :

Given

$$u = \frac{ax - by}{x^2 + y^2}, \; v = \frac{ay + bx}{x^2 + y^2}, \; w = 0. \tag{1}$$

From (1),

$$\frac{\partial u}{\partial x} = \frac{a(x^2 + y^2) - 2x(ax - by)}{(x^2 + y^2)^2} = \frac{ay^2 - ax^2 + 2bxy}{(x^2 + y^2)^2}$$

and

$$\frac{\partial v}{\partial y} = \frac{a(x^2 + y^2) - 2y(ay + bx)}{(x^2 + y^2)^2} = \frac{ax^2 - ay^2 - 2bxy}{(x^2 + y^2)^2}.$$

We see that

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

and hence the equation of continuity is satisfied by (1). Therefore (1) represents a two-dimensional motion and hence vorticity components are given by

$$\Omega_x = 0, \; \Omega_y = 0, \; \Omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}. \tag{2}$$

From (1),

$$\frac{\partial u}{\partial y} = \frac{-b(x^2+y^2)-2y(ax-by)}{(x^2+y^2)^2} = \frac{by^2-bx^2-2axy}{(x^2+y^2)^2}$$

and

$$\frac{\partial v}{\partial x} = \frac{b(x^2+y^2)-2x(ay+bx)}{(x^2+y^2)^2} = \frac{by^2-bx^2-2axy}{(x^2+y^2)^2}$$

so that $\Omega_z = 0$. Thus

$$\Omega_x = 0,\ \Omega_y = 0,\ \Omega_z = 0$$

showing that the motion is irrotational.

Example 2

Find the necessary and sufficient conditions that vortex lines may be at right angles to the streamlines.

Solution :

Streamlines and vortex lines are given by

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} \tag{1}$$

and

$$\frac{dx}{\Omega_x} = \frac{dy}{\Omega_y} = \frac{dz}{\Omega_z} \tag{2}$$

respectively. These will be at right angles, if

$$u\Omega_x = v\Omega_y = w\Omega_z. \tag{3}$$

But

$$\Omega_x = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z},\ \Omega_y = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x},\ \Omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}. \tag{4}$$

Using (4), (3) may be written as

$$u\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + v\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + w\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = 0, \tag{5}$$

which is the necessary and sufficient condition that udx + vdy + wdz may be a perfect differential. So we may write

$$udx + vdy + wdz = \mu d\phi = \mu\left(\frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz\right).$$

Thus the necessary and sufficient conditions that vortex lines may be at right angles to the streamlines are

$$u = \mu\frac{\partial\phi}{\partial x},\ v = \mu\frac{\partial\phi}{\partial y}, w = \mu\frac{\partial\phi}{\partial z}.$$

Example 3

When an infinite liquid contains two parallel, equal and opposite rectilinear vortices at a distance 2b, prove that the streamlines relative to this system are given by the equation

$$\log\frac{x^2 + (y-b)^2}{x^2 + (y+b)^2} + \frac{y}{b} = C,$$

the origin being the midpoint of the line joining the two vortices, taken as the y-axis.

Solution :

Let there be two rectilinear vortices of strengths k and – k at $P_1(0, b)$ and $P_2(0, -b)$ respectively. Thus $P_1P_2 = 2b$, origin being the midpoint of P_1P_2 and y-axis being taken along P_1P_2. Thus we have a vortex pair which will move with a uniform velocity $k/2\pi P_1P_2$ or $k/4\pi b$ perpendicular to the line P_1P_2 (ie. along the x-axis). To determine the streamlines relative to the vortices, we must impose a velocity on the given system equal and opposite to the velocity $k/4\pi b$ of motion of the vortex pair. Accordingly, we add a term $\frac{kz}{4\pi b}$ to the complex potential of the vortex pair. Note that

$$-\frac{d}{dz}\left(\frac{kz}{4\pi b}\right),$$

and hence the term added is justified. So, for the case under consideration, the complex potential is given by

$$w = \phi + i\psi = \frac{ik}{2\pi}\log(z - ib) - \frac{ik}{2\pi}\log(z + ib) + \frac{kz}{4\pi b}.$$

Equating the imaginary parts, we have

$$\psi = \frac{k}{4\pi}\log\left[x^2 + (y-b)^2\right] - \frac{k}{4\pi}\log\left[x^2 + (y+b)^2\right] + \frac{kz}{4\pi b}$$

or

$$\psi = \frac{k}{4\pi}\left[\log\frac{x^2+(y-b)^2}{x^2+(y+b)^2}+\frac{y}{b}\right].$$

Hence the required relative streamlines are given by ψ = constant, i.e.,

$$\log\frac{x^2+(y-b)^2}{x^2+(y+b)^2}+\frac{y}{b}=C.$$

Example 4

If n rectilinear vortices of the same strength k are symmetrically arranged as generators of a circular cylinder of radius a in an infinite liquid, prove that the vortices will move round the cylinder uniformly in time $8\pi^2a^2/(n-1)k$, and find the velocity of any part, of the liquid.

Solution :

Let us take the origin as the center of the circle of radius a and the x-axis along the line $\theta = 0$. Suppose that n rectilinear vortices each of strength k be situated at points $z_m = a\exp^{2\pi im/n}$, m = 0, 1, 2, ..., n – 1 on the circumference of the circle. Then the complex potential due to these n vortices is given by

$$w=\frac{ik}{2\pi}\sum_{m=0}^{n-1}\log(z-a\exp^{2\pi im/n})$$

$$=\frac{ik}{2\pi}\prod_{m=0}^{n-1}(z-a\exp^{2\pi im/n})=\frac{ik}{2\pi}\log(z^n-a^n).$$

Now the fluid velocity q at any point out of all the n vortices is given by

$$q=\left|-\frac{dw}{dz}\right|=\left|\frac{ik}{2\pi}\frac{z^{n-1}}{z^n-a^n}\right|=\left|\frac{kn}{2\pi}\frac{z^n-1}{z^n-a^n}\right|.$$

Again the velocity induced at the point z = a, by the other vortices is given by the complex potential

$$w'=\frac{ik}{2\pi}\log(z^n-a^n)-\frac{ik}{2\pi}\log(z-a)$$

so that

$$w - \frac{dw}{dz} = \frac{ik}{2\pi}\log(z^{n-1} + z^{n-1}a + \cdots + za^{n-1} + a^{n-1}).$$

Hence

$$\left(\frac{dw'}{dz}\right)_{z=a} = \frac{ik}{2\pi}\frac{(n-1)+(n-2)+\cdots+2+1}{na} = \frac{ik(n-1)}{4\pi a}$$

or

$$u_1 - iv_1 = \left(\frac{dw'}{dz}\right)_{z=a} = -\frac{ik(n-1)}{4\pi a}$$

so that $u_1 = 0$ and $v_1 = \frac{k(n-1)}{4\pi a}$. If q_r and q_θ be the radial and transverse velocity components of the velocity at z = a, then we have $q_r = 0$ and $q_\theta = \frac{k(n-1)}{4\pi a}$. Due to symmetry of the problem, it follows that each vortex moves with the same transverse velocity $\frac{k(n-1)}{4\pi a}$. Hence the required time T is given by

$$T = \frac{2a\pi}{\frac{k(n-1)}{4\pi a}} = \frac{8\pi^2 a^2}{(n-1)k}.$$

3.6 Model Questions

Short Questions :

1. Define : Vortex (or vortex filament), vortex lines, vortex tubes, rectilinear vortex, circular vortex, vortex pair, vortex doublet.
2. Prove the following results :
   - (a) Vortex lines and tubes move with the fluid.
   - (b) Strength of a vortex tube is constant alongth the length and for all time.
   - (c) Vortex lines and tubes cannot originate or terminate at internal points in a fluid.
3. Find the expression for the angular velocity of a pair of vortices.

4. Show that the motion due to a circular cylinder is the same as that due to a suitable vortex doublet placed at the centre, with its axis perpendicular to the direction of motion.

Broad Questions :

1. Find the complex potential due to n vortices of strengths $k_1, k_2, ..., k_n$. Hence find the velocity components of the vortex of strength k_s $(1 \leq s \leq n)$. Also, show that the centre of gravity of the vortex system remains at rest.
2. Discuss the motion of an infinite row of vortices, each of strength k situated in a straight line at equal distance apart. Hence show that, at a great distance from the row, the stream lines are parallel to the row.
3. What is meant by Karman Vortex street? Discuss the motion of rectilinear vortices lying on such a street. Also deduce the condition of stability of Karman Vortex street.

Problems :

1. In example1 find the velocity potential of the system.
2. If $udx + vdy + wdz = d\theta + \lambda d\chi$, where θ, λ, χ are function of x, y, z, t, prove that the the vortex lines at any time are the lines of intersection of the surfaces

$$\lambda = \text{constant and } \chi = \text{constant.}$$

3. If in the solved example-3, the vortices are of the same strength and the spin is in same sense both, show that the relative streamlines are given by

$$\log(r^4 + b^4 - 2b^2r^2\cos 2\theta) - (r^2/2b)^2 = \text{constant},$$

θ being measured from the join of the vortices, the origin being its middle point. Show also that the surfaces of equipressure at any instant are given by $r^4 + b^4 - 2b^2r^2\cos 2\theta = \lambda(r^2 \cos 2\theta + a^2)$.

4. Three parallel rectilinear vortices of the same strength K and in the same sense meet any plane perpendicular to them in an equilateral triangle of side a. Show that the vortices all move round the same cylinder with uniform speed in time $\frac{2\pi a^2}{3K}$.

5. If (r_1, θ_1), (r_2, θ_2), ..., be polar coordinates at time t of a system of rectilinear vortices of strength k_1, k_2, ..., prove that

$$\sum kr^2 = \text{constant and } \sum kr^2\dot{\theta} = (1/2\pi)\sum k_1 k_2 .$$

6. An infinite row of equidistant rectilinear vortices are at a distance a apart. The vortices are of the same numerical strength k but they are alternately of opposite signs. Find the complex function that determines the velocity potential and the stream function. Show also that, if α be the radius of a vortex, the amount of flow between two vortex and the next is $(k/\pi)\log\cot(\pi\alpha/2a)$.

7. An infinite street of linear parallel vortices is given as : $x = ra$, $y = b$, strength k;$x = ra$, $y = -b$, strength $= -k$, where r is any positive or negative integer or zero. Prove that if the liquid at infinity is at rest, the street moves as a whole in the direction of its length with the speed $(k/2a)\coth(2\pi b/a)$.

Unit 4 □ Surface Waves

Structure

4.0 Introduction

4.1 General expression for wave motion

4.2 Wave motion in liquid

4.3 Standing or Stationary Waves

4.4 Surface Waves

4.4.1 Progressive Waves on the surface of water

4.4.2 Progressive Waves on a deep water

4.4.3 Stationary Waves on the surface of water

4.5 The Energy of the progressive waves

4.6 Group Velocity

4.7 Rate of transmission of energy in simple harmonic surface waves

4.8 Progressive waves reduced to a case of steady motion

4.9 Waves at the common surface of two liquids

4.9.1 Waves at the interface of two liquids with upper surface free

4.10 Long waves of small elevation

4.11 Capillary Waves

4.11.1 Capillary waves in a channel of uniform depth

4.12 Examples

4.13 Model Questions

4.14 Summary

4.0 Introduction

It is a matter of common observation that if a pebble is thrown into a pond, then some disturbance travels radially over the water surface. Such a disturbance is known as water waves. Also, if a piano is played in a room, then sound wave is spread there. The energy extracted from the sun is transmitted through waves in ether. All these are examples of wave motion. Thus we nitice two distinguished features : (a) *energy is propagated at distant points* and (b) *the disturbance travels through the medium without any transference of the medium itself.* In fact, these two properties do exist whatever be the medium which transmits the waves.

4.1 General Expression for Wave Motion

Consider an arbitrary disturbance ϕ moving along the positive x-axis with velocity c. Thus ϕ is a function of x and t, say $\phi = f(x, t)$. The curve when $t = 0$, i.e., $\phi = f(x)$ is known as *wave profile.* If the disturbance moves without changing its shape, then the wave profile has moved through a distance ct in the positive direction of x-axis at time t. If the distance measured from the new origin $x = ct$ be denoted by ξ so that $x - ct = \xi$, then the equation of the wave profile referred to the new origin is $\phi = f(\xi)$, in other words, referred to the original origin, it is

$$\phi = f(x - ct). \tag{1}$$

Similarly, the equation $\phi = f(x + ct)$ represents the same disturbances moving in the negative direction of x-axis with velocity c.

4.2 Wave Motion in Liquid

A wave motion of a liquid acted upon by gravity and having a free surface is a motion in which the elevation of the free surface above some chosen fixed horizontal plane varies.

Taking the axis of x to be horizontal and the axis of y to be vertically upwards, a motion in which the equation of the vertical section of the free surface at time t is of the form

$$y = a \sin (mx - nt), \tag{2}$$

where a, m, n are constants, is called a Simple harmonic progressive wave. Since (2) can be written in the form

$$y = a \sin m\left(x - \frac{nt}{m} \right), \tag{3}$$

this shows that the wave profile $y = a \sin mx$ at $t = 0$ moves with velocity n/m (= c, say) in the positive x-direction. c is called the *velocity of propagation* of the wave. When

$a = 0$ the profile of the liquid is $y = 0$, which is the *mean level.* The quantity a is called the *amplitude* of the wave and measures the maximum departure of the actual free surface from the mean level. The points $C_1, C_2, \ldots$ of maximum elevation are known as *crests* and the points $T_1, T_2, \ldots$ of maximum depression are known as *troughs.* The distance between successive crests is called the *wave-length* and is denoted by λ. Thus

$$\lambda = \frac{2\pi}{m}.$$

Again the nature of the free surface (2) remains unchanged by replacing t by $t + 2\pi/n$. The time $T = 2\pi/n$ is known as the *period* of the wave. The reciprocal of the period is known as the *frequency* it denotes the number of oscillations per second. The angle $mx - nt$ is known as *phase angle.* If the equation of wave motion be $y = a\sin(mx - nt + \epsilon)$, then ϵ is called the *phase* of the wave.

4.3 Standing or Stationary Waves

Two simple harmonic progressive waves of the same amplitude, wave length and period travel in opposite directions are given by the surface elevation

$$\eta_1 = \frac{1}{2} a\sin(mx - nt), \ \eta_2 = \frac{1}{2} a\sin(mx + nt).$$

By the principle of superposition, the resulting surface elevation is represented by the equation

$$\eta = \eta_1 + \eta_2 = 2a\sin mx\cos nt.$$

A motion of this type is called a *stationary or standing wave.* An any instant the equation represents a sine curve but the amplitude $2a\cos nt$ varies continuously.

The points of intersection of the curve with the x-axis are fixed points called *nodes.* When a progressive train of waves represented by η_1 impinges on a fixed vertical barrier and is there reflected (η_2), the resulting disturbance when a steady state is reached consists of stationary waves.

Such waves can, for example, be generated by tilting slightly a rectangular vessel containing water and then restoring it to the level position. The water level at each end of the vessel then moves up and down the vertical faces which are loops. Conversely a progressive wave can be regarded as due to the superposition of two standing waves.

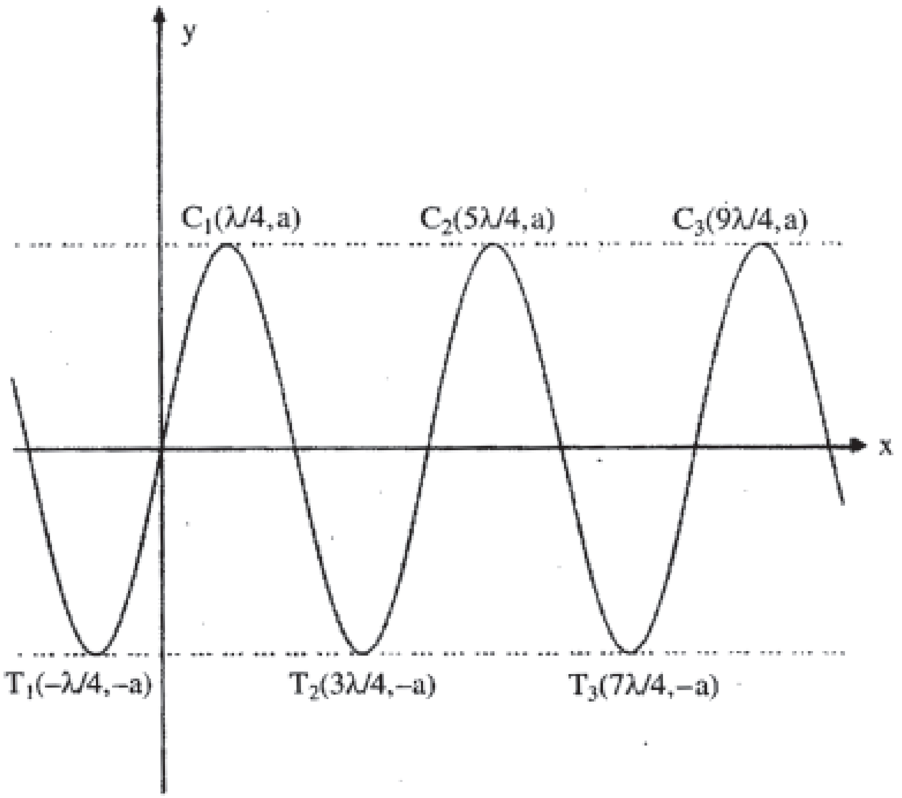

Figure 4.1

4.4 Surface Waves

Such waves occur at and near the free surface of an unbounded sheet of liquid where the depth is considerable compared to the wave length. For these waves the vertical acceleration is comparable with the horizontal acceleration, and so we consider forces both in horizontal and vertical directions.

The x-axis is taken in the undisturbed surface in the direction of propagation of the waves and the y-axis vertically upwards. Taking the motion to be irrotational, incompressible and two-dimensional, the velocity potential ϕ exits such that

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \qquad (4)$$

throughout the liquid, and

$$\frac{\partial\phi}{\partial n} = 0 \tag{5}$$

at a fixed boundary.

The pressure can be obtained from the Bernoulli's equation

$$\frac{p}{\rho} = \frac{\partial\phi}{\partial t} - gy - \frac{1}{2}q^2 + C(t). \tag{6}$$

The free surface is a surface of equipressure p = constant, hence on the free surface

$$\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y} = 0, \tag{7}$$

where u and v are the velocity components on the free surface in x and y directions respectively. But

$$u = -\frac{\partial\phi}{\partial x}, \; v = -\frac{\partial\phi}{\partial y}, \tag{8}$$

and at the free surface the relation (7) becomes

$$\frac{\partial p}{\partial t} - \frac{\partial\phi}{\partial x}\frac{\partial p}{\partial x} - \frac{\partial\phi}{\partial y}\frac{\partial p}{\partial y} = 0. \tag{9}$$

Let the motion be so small that the squares of small quantities may be omitted. Again, without loss of generality we may include C(t) in ϕ and then substitute the value of p from (6) in (9) to get

$$\frac{\partial^2\phi}{\partial t^2} - \frac{\partial\phi}{\partial x}\frac{\partial^2\phi}{\partial x\partial t} - \frac{\partial\phi}{\partial y}\left(\frac{\partial^2\phi}{\partial y\partial t} - g\right) = 0. \tag{10}$$

Neglecting the second and third terms which are of the same order as q^2, we obtain

$$\frac{\partial^2\phi}{\partial t^2} + g\frac{\partial\phi}{\partial y} = 0. \tag{11}$$

This condition holds at the free surface.

If η is the elevation of the free surface at time t above the point whose abscissa is x, the equation of the free surface is given by

$$y - \eta(x, t) = 0. \tag{12}$$

But we know that if

$$F(x, y, t) = y - \eta(x, t) = 0$$

is a boundary surface, then we must have

$$\frac{\partial F}{\partial t} + u\frac{\partial F}{\partial x} + v\frac{\partial F}{\partial y} = 0$$

or

$$\frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} - v = 0. \tag{13}$$

But $\frac{\partial \eta}{\partial t}$ is $\dot{\eta}$, and $\frac{\partial \eta}{\partial x}$ is the tangent of the slope of the free surface which by hypothesis is small so that the second term can be neglected and the equation becomes

$$\dot{\eta} = v = -\frac{\partial \phi}{\partial y} \tag{14}$$

at the free surface.

Hence in a wave motion in which the squares of the velocities can be neglected, the velocity potential must be a solution of Laplace's equation which makes

$$\frac{\partial \phi}{\partial n} = 0$$

at a fixed boundary and satisfies (11) and (14) at the free surface of the liquid.

4.4.1 Progressive waves on the surface of water

Consider the propagation of simple harmonic waves of the type

$$\eta = a\sin(mx - nt) \tag{15}$$

at the surface of water of uniform depth h, either of unlimited extent or contained in a channel with parallel vertical sides at right angle to the ridges and hollows.

If we assume that there is a solution of the form

$$\phi = f(y)\cos(mx - nt)$$

and substitute in (4) we obtain

$$\frac{\partial^2 f}{\partial y^2} - m^2 f = 0, \tag{16}$$

so that

$$f(y) = Ae^{my} + Be^{-my},$$

and

$$\phi = (Ae^{my} + Be^{-my}) \cos(mx - nt).$$

This value of f must be satisfy (15), i.e.

$$\frac{\partial \phi}{\partial y} = 0 \qquad \text{when } y = -h.$$

Hence

$$Ae^{-mh} = Be^{mh} = \frac{1}{2}C, \quad \text{say},$$

so that

$$\phi = C \cosh m(y + h) \cos(mx - nt). \tag{17}$$

Again if we substitute this value in the surface condition (8) putting $y = 0$, we get

$$n^2 = gm \tanh mh. \tag{18}$$

Now let $c = n/m$ and $\lambda = 2\pi/m$ denote velocity of propagation and the wave length respectively. Then we get

$$c^2 = \frac{g}{m} \tanh mh = \frac{g\lambda}{2\pi} \tanh \frac{2\pi h}{\lambda}. \tag{19}$$

We now determine the constant C of (17) in terms of the amplitude a of the wave. Using (15) and (17), the boundary condition (14) gives

$$-na = -mC \sinh mh,$$

so that

$$\phi = \frac{na}{m} \frac{\cosh m(y + h)}{\sinh mh} \cos(mx - nt), \tag{20}$$

or, using (18) we obtain

$$\phi = \frac{ga}{n} \frac{\cosh m(y + h)}{\cosh mh} \cos(mx - nt). \tag{21}$$

The path of the particle

If (x, y) be the coordinates of a particle relative to its mean position, neglecting the squares of small quantities we may write

$$\frac{dx}{dt} = -\frac{\partial\phi}{\partial x} = na\frac{\cosh m(y+h)}{\sinh mh}\sin(mx - nt),$$

$$\frac{dy}{dt} = -\frac{\partial\phi}{\partial y} = -na\frac{\sinh m(y+h)}{\sinh mh}\cos(mx - nt).$$

Integrating above two equations, we get

$$x = a\frac{\cosh m(y+h)}{\sinh mh}\cos(mx - nt),$$

$$y = a\frac{\sinh m(y+h)}{\sinh mh}\sin(mx - nt);$$

so that the particle describes the ellipse

$$\frac{x^2}{\cosh^2 m(y+h)} + \frac{y^2}{\sinh^2 m(y+h)} = \frac{a^2}{\sinh^2 mh}$$

about its mean position. For a given particle mx – nt plays the part of the eccentric angle in the ellipse; so that the eccentric angle increases at a uniform rate, as in an orbit described under a central force varying as the distance.

4.4.2 Progressive waves on a deep water

If the depth h of the water be sufficiently great in comparison with λ for e^{-mh} to be neglected, then the constant B = 0 in the above case, so that we have instead of (17)

$$\phi = Ae^{my}\cos(mx - nt) \qquad (22)$$

and instead of (18)

$$n^2 = gm \qquad (23)$$

or,

$$c^2 = \frac{g\lambda}{2\pi}. \qquad (24)$$

Also if

$$\eta = a \sin(mx - nt)$$

is the free surface we get from (14)

$$na = mA,$$

so that

$$\phi = \frac{na}{m} e^{my} \cos(mx - nt),$$

$$\phi = \frac{ga}{n} e^{my} \cos(mx - nt). \tag{25}$$

Following the case **4.4.1** we get in this case for the displacement of a particle from its mean position

$$\begin{aligned} x &= ae^{my} \cos(mx - nt), \\ y &= ae^{my} \sin(mx - nt), \end{aligned} \tag{26}$$

and the path of the particle is a circle

$$x^2 + y^2 = a^2 e^{2my},$$

described with uniform angular velocity n, which in this case is equal to $(gm)^{\frac{1}{2}}$ or $\left(\frac{2\pi g}{\lambda}\right)^{\frac{1}{2}}$.

4.4.3 Stationary waves on the surface of water

Consider a stationary wave of the type

$$\eta = a \sin mx \cos nt. \tag{27}$$

The velocity potential for a system of stationary waves can be deduced from **4.4.1** by regarding the system as the result of the superposition of two such trains of waves as we have just been considered moving in opposite directions as explained in **Section-4.3.**

Then we shall have

$$\phi = \frac{na}{m} \frac{\cosh m(y + h)}{\sinh mh} \sin mx \sin nt, \tag{28}$$

or,

$$\phi = \frac{ga}{n}\frac{\cosh m(y+h)}{\cosh mh}\sin mx \sin nt \qquad (29)$$

for ϕ satisfies (4) and (5), and η and ϕ together satisfy (14).

It is not necessary to regard standing waves as a case of superpositon of progressive waves. We might investigate this form for ϕ independently starting with the assumption

$$\phi = f(y) \sin mx \sin nt.$$

For standing waves in deep water, as in **4.4.2,** equations (28) and (29) take the forms

$$\left.\begin{aligned} \phi &= \frac{na}{m} e^{my} \sin mx \sin nt, \\ \phi &= \frac{ga}{n} e^{my} \sin mx \sin nt. \end{aligned}\right\} \qquad (30)$$

Path of the particles :

In this case we have

$$\dot{x} = -\frac{\partial\phi}{\partial x} = -na\frac{\cosh m(y+h)}{\sinh mh}\cos mx \sin nt,$$

$$\dot{y} = -\frac{\partial\phi}{\partial y} = -na\frac{\sinh m(y+h)}{\sinh mh}\cos mx \sin nt,$$

so that, by integration

$$x = a\frac{\cosh m(y+h)}{\sinh mh}\cos mx \cos nt,$$

and

$$y = a\frac{\sinh m(y+h)}{\sinh mh}\sin mx \cos nt.$$

Hence

$$\frac{y}{x} = \tanh m(y+h)\tan mx,$$

and since this is independent of t, the motion of each particle is rectilinear, the direction varying from vertical beneath the crests and troughs $\left(mx = (n+\frac{1}{2})\pi\right)$, to horizontal beneath the nodes ($mx = n\pi$).

4.5 The Energy of the Progressive Waves

Kinetic Energy **:** The kinetic energy possessed by the liquid (per unit thickness), stretching between two vertical plane situated at a distance of one wave length apart and perpendicular to the direction of flow, is known as the kinetic energy of the progressive wave.

Considering a train of progressive waves at the surface of water of depth h, given by

$$\eta = a \sin(mx - nt) \tag{31}$$

and

$$\phi = \frac{ga}{n} \frac{\cosh m(y+h)}{\cosh mh} \cos(mx - nt). \tag{32}$$

Since the motion is irrotational, the kinetic energy is given by

$$T = -\frac{1}{2}\rho \int \phi \frac{\partial \phi}{\partial n} ds, \tag{33}$$

δn being normal drawn into the liquid and integration being performed along the profile of a wave length. In this case, we get kinetic energy

$$T = \frac{1}{2}\rho \int_0^\lambda \left[\phi \frac{\partial \phi}{\partial y} \right]_{y=0} dx$$

$$= \frac{1}{2}\rho g a^2 \int_0^\lambda \cos^2 (mx - nt)\, dx$$

$$= \frac{1}{2}\rho g a^2 \lambda .$$

Potential Energy :

The potential energy due to the elevated liquid in a wave length (the energy being calculated relative to the undisturbed state) is known as the potential energy of a progressive wave.

Let us calculate the potential energy of liquid between two vertical planes parallel to the direction of propagation at unit distance apart. Then, for a single wave length, the potential energy is given by

$$V = \frac{1}{2}\rho\int_0^\lambda \eta^2 \, dx$$

$$= \frac{1}{4}\rho g a^2 \lambda,$$

as $\lambda = 2\pi/m$.

Total energy per wave length is

$$= T + V$$

$$= \frac{1}{2}\rho g a^2 \lambda.$$

Hence it follows that the total energy per wave length is half kinetic energy and potential energy.

The energy of the stationary waves :

The energy of stationary waves may be calculated in the same way. Thus if we take

$$\eta = a \sin mx \cos nt$$

and

$$\phi = \frac{ga}{n} \frac{\cosh m(y+h)}{\cosh mh} \sin mx \sin nt.$$

We find for the potential energy of a wave length

$$V = \frac{1}{4} g a^2 \rho \lambda \cos^2 nt,$$

and for the kinetic energy

$$T = \frac{1}{4} g a^2 \rho \lambda \sin^2 nt.$$

Total energy per wave length at any time

$$= T + V$$

$$= \frac{1}{4} g a^2 \rho \lambda.$$

The amounts of kinetic and potential energy change continuously with the time.

4.6 Group Velocity

A local disturbance of the surface of still water will give rise to a wave which can be analyzed into a set of simple harmonic components each of different wave-length. We have seen that the velocity of propagation depends upon the wave-length and so the waves of different wave-lengths will be gradually sorted out into groups of waves of approximately the same wave-length. In the case of water waves, the velocity of the group is, in general, less than the velocity of the individual waves composing it. What happens in this case is that the waves in front pass out of the group and new waves enter the group from behind. The energy within the group remains the same.

We now study the properties of such a group. To this end we examine the disturbance due to the superposition of two simple harmonic waves of the same amplitude and slightly different wave lengths,

$$\eta_1 = a \sin(mx - nt),$$

$$\eta_2 = a \sin\{(m + \delta m)x - (n + \delta n)t\}.$$

The resulting disturbance will be

$$\eta = \eta_1 + \eta_2$$

$$= 2a\cos\frac{1}{2}(x\delta m - t\delta n)\sin(mx - nt)$$

$$= A\sin(mx - nt) \qquad (34)$$

where

$$A = 2a\cos\frac{1}{2}(x\delta m - t\delta n). \qquad (35)$$

Equation (34) shows that the resulting disturbance is a progressive sine wave whose amplitude A is not constant but is itself varying as a wave of velocity

$$c_g = \frac{\delta n}{\delta m}. \qquad (36)$$

This velocity is known as the *group velocity*.

Since the velocity of propagation of a single wave is

$$c = \frac{n}{m},$$

we have

$$c_g = \frac{dn}{dm} = \frac{d}{dm}(cm) = c + m\frac{dc}{dm}. \tag{37}$$

But $\lambda = \frac{2\pi}{m}$ so that

$$\frac{d\lambda}{dm} = -\frac{2\pi}{m^2}. \tag{38}$$

Then we get

$$c_g = c + m\frac{dc}{d\lambda}\frac{d\lambda}{dm} = c - \lambda\frac{dc}{d\lambda}. \tag{39}$$

For the case of waves on the surface of liquid of depth h, we have

$$c^2 = \frac{g}{m}\tanh mh. \tag{40}$$

From (37) and (40), we have

$$\begin{aligned} c_g &= c\left(1 + \frac{m}{2c^2}\frac{dc^2}{dm}\right) \\ &= \frac{1}{2}c\left(1 + \frac{2mh}{\sinh 2mh}\right) \end{aligned} \tag{41}$$

so that the ratio of the group velocity to the wave velocity is given by

$$\frac{c_g}{c} = \frac{1}{2} + \frac{mh}{\sinh 2mh}$$

or,

$$c_g = \frac{1}{2}c\left(1 + \frac{2mh}{\sinh 2mh}\right). \tag{42}$$

When h is small compared with the wave length, this ratio is unity, so that group velocity for shallow water is equal to the wave velocity. Also as h increases to infinity the ratio decreases to $\frac{1}{2}$; or the group velocity for deep sea waves is half the wave velocity.

4.7 Rate of Transmission of Energy in Simple Harmonic Surface Waves

In a simple harmonic train of surface waves, energy crosses a fixed vertical plane perpendicular to the direction of propagation at an average rate equal to group velocity.

Proof :

Consider vertical section of the liquid at right angle to the direction of propagation. Then the rate of transmission of energy is calculated by determining the rate at which the pressure on one side of the chosen section is doing work on the liquid on the other side. Now, the velocity potential is given by

$$\phi = \frac{ga}{n}\frac{\cosh m(y+h)}{\cosh mh}\cos(mx - nt). \qquad (43)$$

Again neglecting squares of small quantities the variable part of the pressure is given by

$$\delta p = \rho\frac{\partial \phi}{\partial t}. \qquad (44)$$

and the horizontal velocitv is

$$u = -\frac{\partial \phi}{\partial x}. \qquad (45)$$

Hence the work done in unit time or the energy carried across unit width of the section is

$$W = -\int_{-h}^{0} \delta p \frac{\partial \phi}{\partial x} dy$$

$$= \frac{g^2 a^2 \rho m}{n}\frac{\sin^2(mx - nt)}{\cosh^2 mh}\int_{-h}^{0}\cosh^2 m(y+h)\,dy$$

$$= \frac{g^2 a^2 \rho m}{n}\frac{\sin^2(mx - nt)}{\cosh^2 mh}\left(\frac{\sinh 2mh}{4m} + \frac{h}{2}\right), \qquad (46)$$

and since

$$n^2 = gm \tanh mh,$$

this may be written as

$$W = \frac{1}{2} g\rho a^2 \frac{n}{m}\left(1 + \frac{2\,mh}{\sinh 2\,mh}\right)\sin^2(mx - nt). \tag{47}$$

The mean value of the expression (47) over a complete period or any number of complete periods, or any interval that is so long compared to a period that the part corresponding to the fractional part of a period can be neglected in comparison with the whole, is

$$W = \frac{1}{4} g\rho a^2 \frac{n}{m}\left(1 + \frac{2\,mh}{\sinh 2\,mh}\right). \tag{48}$$

But the group velocity c_g is given by

$$c_g = \frac{1}{2} c\left(1 + \frac{2\,mh}{\sinh 2\,mh}\right). \tag{49}$$

Since $\frac{n}{m} = c$, then from (48) and (49) we get

$$W = \frac{1}{2}\left(\frac{1}{2} g\rho a^2\right) c_g. \tag{50}$$

Since $\frac{1}{2} g\rho a^2$ is the whole energy per unit length at any instant. Hence the energy is transmitted at a rate equal to the group velocity.

4.8 Progressive Waves Reduced to a Case of Steady Motion

In any case in which waves propagate in one direction only without change of shape, the problem of determining the velocity of propagation can be simplified as follows : Impose on the whole liquid a velocity equal and opposite to the velocity of propagation of the waves. Then the wave profile having the same relative velocity as before becomes fixed in space and the problem becomes one of steady motion. We now illustrate this technique by means of the following two cases :

Case-I : *Progressive waves on the surface of water*

Let progressive waves move on the surface on the channel of uniform depth h and having parallel vertical walls. Let the progressive waves move towards the positive direction of x-axis with velocity c without change of form. Impose on the whole liquid a velocity c in the negative direction of x-axis. The wave form having the same relative velocity as before becomes fixed in space and the problem becomes one of steady motion. As the problem is a two-dimensional one it only remains to determine suitable expressions for the velocity potential and stream function so that the free surface and the bottom of the liquid may satisfy the conditions for stream lines.

Consider the complex potential

$$w = cz + P \cos mz - iQ \sin mz,$$

or,

$$\phi + i\psi = c(x + iy) + P\cos m(x + iy) - iQ\sin m(x + iy).$$

It gives

$$\left.\begin{aligned} \phi &= cx + (P\cosh my + Q\sinh my)\cos mx, \\ \psi &= cy - (P\sinh my + Q\cosh my)\sin mx. \end{aligned}\right\} \tag{51}$$

Since ϕ and ψ given by (51) satisfy Laplace's equation, they represent a possible motion.

For the bottom to be a stream line we must have ψ is constant when $y = -h$ so that

$$-P \sinh mh + Q \cosh mh = 0.$$

Hence the expressions (51) may be written as

$$\left.\begin{aligned} \phi &= cx + A\cosh m(y + h)\cos mx, \\ \psi &= cy - A\sinh m(y + h)\sin mx. \end{aligned}\right\} \tag{52}$$

Let the free surface be a simple curve

$$\eta = a \sin mx.$$

Then from (4) the stream line $\psi = 0$ produces

$$ca - A \sinh mh = 0. \tag{53}$$

neglecting squares of small quantities.

Again, the formula for pressure is

$$\frac{p}{\rho} + gy + \frac{1}{2}\left\{\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2\right\} = \text{constant}. \tag{54}$$

At the free surface

$$y = a \sin mx$$

this becomes

$$\frac{p}{\rho} + ga \sin mx + \frac{1}{2} c^2 \{1 - 2\, ma \coth mh \sin mx\} = \text{constant}, \tag{55}$$

neglecting a^2.

But p is constant at the free surface. Hence (55) holds if the coefficient of sin mx vanishes, i.e.

$$g = c^2 m \coth mh,$$

or,

$$c^2 = \frac{g\lambda}{2\pi} \tanh \frac{2\pi h}{\lambda}. \tag{56}$$

Case-II : *Progressive waves on a deep water*

For this case (when $h \to \infty$) we consider

$$\phi = cx + Ae^{my} \cos mx, \tag{57}$$

and

$$\psi = cy - Ae^{my} \sin mx \tag{58}$$

with a free surface

$$\eta = a \sin mx. \tag{59}$$

The free surface is the stream line $\psi = 0$, if

$$ca = A, \tag{60}$$

so that

and
$$\left.\begin{aligned} \phi &= cx + cae^{my} \cos mx, \\ \psi &= cy - cae^{my} \sin mx. \end{aligned}\right\} \tag{61}$$

The formula for the pressure

$$\frac{p}{\rho} + gy + \frac{1}{2}\left\{\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2\right\} = \text{constant}$$

becomes

$$\frac{p}{\rho}+gy+\frac{1}{2}c^2\{1-2\,mae^{my}\sin mx+m^2a^2e^{2my}\}=\text{constant.} \tag{62}$$

If we neglect the last term on the left, this equation may be written as

$$\frac{p}{\rho}+y(g-mc^2)+mc\psi=\text{constant.} \tag{63}$$

This equation not only gives

$$c^2=\frac{g}{m} \tag{64}$$

at the free surface, but also shows that, if $c^2=\frac{g}{m}$, the pressure is constant along each stream line. It follows that the solution contained in (56) and (64) can be applied to the case of any number of liquids of different densities arranged one above the other in horizontal strata including the case of liquid of continuously varying density since there is no limit to the thinness of a stream, the only limitations being that the upper surface is free and the total depth infinite.

4.9 Waves at the Common Surface of Two Liquids

Suppose a liquid of density ρ' and depth h' to be moving with velocity V' over another liquid of density ρ and depth h moving in the same direction with velocity V, the liquids being bounded above and below by two fixed horizontal planes.

Let c be the velocity of propagation of oscillatory waves at the interface of the two liquids in the direction in which the liquids are moving. Let the x-axis be in this direction in the undisturbed interface and y-axis vertically upwards. Let us make the motion steady by superposing on the whole mass the velocity $-c$ thereby bringing the wave form to rest in space.

The velocity and stream function for the lower liquid moving with the velocity $-(V-c)$ in the negative direction of x-axis and given by

$$\left.\begin{aligned}\phi&=-(V-c)x+A\cosh m(y+h)\cos mx,\\ \psi&=-(V-c)x-A\sinh m(y+h)\sin mx.\end{aligned}\right\} \tag{65}$$

Similarly expression for upper liquid may be deduced from (65) by replacing V by V′ and h by − h′. Thus we get

$$\left.\begin{aligned} \phi' &= -(V'-c)x + A'\cosh m(y-h')\cos mx, \\ \psi' &= -(V'-c)x - A'\sinh m(y-h')\sin mx. \end{aligned}\right\} \tag{66}$$

These expression for ψ and ψ' clearly make the boundaries $y = -h$, $y = h'$ stream lines; and if $\eta = a \sin mx$ gives the displacement of the common surface and the liquids do not separate this must be a stream line for both surfaces. We can satisfy this condition by taking the stream line to be $\psi = \psi' = 0$, which gives

$$\left.\begin{aligned} -(V-c)a - A\sinh mh &= 0, \\ -(V'-c)a + A'\sinh mh' &= 0, \end{aligned}\right\} \tag{67}$$

neglecting the squares of small quantities.

From Bernoulli's equations, we obtain

$$\left.\begin{aligned} \frac{p}{\rho} + gy + \frac{1}{2}\left\{\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2\right\} &= \text{constant}, \\ \frac{p'}{\rho'} + gy + \frac{1}{2}\left\{\left(\frac{\partial\phi'}{\partial x}\right)^2 + \left(\frac{\partial\phi'}{\partial y}\right)^2\right\} &= \text{constant}. \end{aligned}\right\} \tag{68}$$

But at the interface

$$y = \eta = a \sin mx.$$

Then (68) gives (neglecting a^2)

$$\left.\begin{aligned} \frac{p}{\rho} + ga\sin mx + \frac{1}{2}(V-c)^2\,\{1 - 2\,ma\coth mh \sin mx\} &= \text{constant}, \\ \frac{p'}{\rho'} + ga\sin mx + \frac{1}{2}(V'-c)^2\,\{1 - 2\,ma\coth mh' \sin mx\} &= \text{constant}. \end{aligned}\right\} \tag{69}$$

Since the pressure is continuous across the interface, putting $p = p'$ in above equations, subtracting and then equating to zero the coefficient of sin mx, we obtain

$$g(\rho - \rho') = (V-c)^2\, m\rho\coth mh + (V'-c)^2\, m\rho'\coth mh'. \tag{70}$$

This equation determines the velocity of propagation c of waves of length $\frac{2\pi}{m}$ at the common surface of two streams whose velocities are V, and V′; but it may also be regarded as the condition for stationary waves at the common surface of two streams whose velocities are V – c and V′ – c.

It should be noticed that in any such case as the above, even when V and V′ are both zero, the tangential velocities on opposite sides of the surface of separation are different so that this surface constitutes a vortex sheet.

4.9.1 Waves at the interface of two liquids with upper surface free

Another case of interest is that in which the surface of the upper liquid is free; *e.g.* a layer of oil upon water or of fresh water upon salt water.

Let a liquid of density ρ′ and depth h′ lie over another liquid of density ρ and depth h and let both the liquids to be at rest save for wave motion. We assume a common velocity of wave propagation c at the free surface of the upper liquid and at the common surface and reverse this velocity on the whole mass so that the motion becomes steady. We may take

$$\psi = cy - A \sinh m(y + h) \sin mx, \tag{71}$$

in the lower liquid, and

$$\psi' = cy - (B \cosh my + C \sinh my) \sin mx \tag{72}$$

in the upper liquid.

From (71), it easily follows that the bottom y = – h is a stream surface ψ = – ch. Let the common surface be given by

$$\eta = a \sin mx, \tag{73}$$

it is also the stream surface ψ = ψ′ = 0, if

$$ca - A \sinh mh = 0, \tag{74}$$

and

$$ca - B = 0. \tag{75}$$

Also the free surface

$$y = h' + b \sin mx \tag{76}$$

is a stream surface ψ′ = constant if

$$cb - (B \cosh mh' + C \sinh mh') = 0. \tag{77}$$

From the Bernoulli's equation for the lower and upper liquids respectively, we have

$$\left.\begin{aligned} \frac{p}{\rho}+gy+\frac{1}{2}\left\{\left(\frac{\partial\psi}{\partial x}\right)^2+\left(\frac{\partial\psi}{\partial y}\right)^2\right\}=\text{constant}, \\ \frac{p'}{\rho'}+gy+\frac{1}{2}\left\{\left(\frac{\partial\psi'}{\partial x}\right)^2+\left(\frac{\partial\psi'}{\partial y}\right)^2\right\}=\text{constant}. \end{aligned}\right\} \tag{78}$$

Substituting from (71) and (72), using that A, B and C are of order a, neglecting squares of small quantities and equating the values of p and p′ at the common interface, we get

$$ga(\rho-\rho')-cm(\rho A\cosh mh-\rho' C)=0, \tag{79}$$

and using (74), (75) and (76), this gives

$$g(\rho-\rho')=c^2 m\left\{\rho\coth mh+\rho'\coth mh'-\rho'\frac{b}{a\sinh mh'}\right\}. \tag{80}$$

Then using the fact that p′ is constant at the free surface we get

$$gb = cm(B\sinh mh' + C\cosh mh'), \tag{81}$$

and from (74), (75) and (77) we obtain

$$g=c^2 m\left(\coth mh'-\frac{a}{b\sinh mh'}\right). \tag{82}$$

The elimination of the ratio a : b from (80) and (82) gives the equation for c, viz.

$$c^4 m^2(\rho\coth mh\coth mh'+\rho')-c^2 m\rho g(\coth mh+\coth mh')+g^2(\rho-\rho')=0 \tag{83}$$

and the ratio of the amplitudes of the waves is given from (82) by

$$\frac{b}{a}=\frac{c^2 m}{c^2 m\cosh mh'-g\sinh mh'}. \tag{84}$$

From (83) we see that there are two possible velocities of propagation for a given wave length, provided $\rho > \rho'$.

In the particular case in which the lower liquid is deep we put coth mh = 1. The roots of (83) are then

$$c^2=\frac{g}{m} \quad\text{and}\quad c^2=\frac{g}{m}\frac{\rho-\rho'}{(\rho\coth mh'-\rho')}.$$

The ratio of the amplitudes of the upper and lower waves in the two cases are

$$e^{mh'} \quad \text{and} \quad -\left(\frac{\rho}{\rho'}-1\right)e^{-mh'}.$$

4.10 Long Waves of Small Elevation

These types of waves arise when the wave length of oscillations is much greater than the depth of the liquid and the disturbance affects the motion of the whole of the liquid.

For simplicity, consider the case of waves travelling in a straight canal of depth y_0 of uniform section. Take the x-axis is the direction of the length of the canal and y-axis vertically upwards and let η be the elevation of the free surface above the equilibrium level at the point whose abscissa is x at time t. We shall consider the case when the wave length λ is large so that (y_0/λ) is very small as well as (η/y_0) and $(d\eta/dx)$.

Then, so far as vertical forces are concerned we may regard the liquid to be in equilibrium and take the pressure at any point as the statical pressure due to the depth below the free surface. Thus the pressure p at a point (x, y) is given by

$$p - p_0 = g\rho(y_0 + \eta - y), \tag{85}$$

where p_0, supposed constant, is the pressure above the liquid. Hence we have

$$\frac{\partial p}{\partial x} = g\rho\frac{\partial \eta}{\partial x} \tag{86}$$

which is independent of y. Thus the horizontal acceleration of an element depends on the difference of pressure at its ends, i.e. $\frac{\partial p}{\partial x}dx$ so that the horizontal acceleration of all points in the same vertical cross-section remains the same. Consequently, those points which are once in a vertical plane always remain there.

We now consider a small horizontal cylinder PP′ of liquid of length dx′ and cross-section α, the difference of pressure at its ends being $g\rho\frac{\partial \eta}{\partial x'}dx'$. Also, if x be the abscissa of the vertical plane of particles through P in its equilibrium position and ξ be the horizontal displacement of this plane of particles, then $x' = x + \xi$ so that the horizontal acceleration is $\frac{\partial^2 \xi}{\partial t^2}$. The equation of motion is, therefore,

$$\rho\alpha dx' \frac{\partial^2 \xi}{\partial t^2} = -g\rho\alpha \frac{\partial \eta}{\partial x'} dx'$$

or,

$$\frac{\partial^2 \xi}{\partial t^2} = -g \frac{\partial \eta}{\partial x'} \tag{87}$$

Assuming the motion to be small and squares y small quantities can be neglected, we have from (87), by putting $x' = x + \xi$

$$\frac{\partial^2 \xi}{\partial t^2} = -g \frac{\partial \eta}{\partial x} \tag{88}$$

To form the equation of continuity, we suppose that A is the area of cross-section of the canal and b is the breadth of the surface. Then, in equilibrium position, the volume of liquid containing between the planes x and x + dx is Adx. Also, at time t, the distance between the bounding planes of the liquid is $dx + \frac{\partial \xi}{\partial x} dx$ and the area of the cross-section is $A + b\eta$. Thus

$$(A + b\eta)\left(dx + \frac{\partial \xi}{\partial x} dx \right) = Adx$$

or,

$$A \frac{\partial \xi}{\partial x} + b\eta = 0 \tag{89}$$

where we have neglected product of small quantities. Thus, from (88) we obtain

$$\frac{\partial^2 \xi}{\partial t^2} = \frac{gA}{b} \frac{\partial^2 \xi}{\partial x^2}$$

whose solution is

$$\xi = f(x - ct) + F(x + ct) \tag{90}$$

where $c^2 = gA/b$. Equation (90) represents two waves travelling in opposite directions with velocity $c = (gA/b)^{\frac{1}{2}}$.

For a canal of rectangular cross-section of depth h, the wave velocity is $(gh)^{\frac{1}{2}}$ which is half the depth of the liquid.

The elevation η is given by (89) and (90) as

$$\eta = -\frac{A}{b} f'(x - ct) - \frac{A}{b} F'(x + ct) \tag{91}$$

Also, the particle velocity is

$$\dot{\xi} = -cf'(x - ct) + cF'(x + ct). \tag{92}$$

4.11 Capillary Waves

Let there be an interface between two liquids, like water in contact with air. This the interface will not be a constant pressure surface unless it is a plane surface. Since free surface is a curved surface, so waves would be effected due to a surface tension or energy per unit area due to capillary forces, the difference of the pressure on opposite sides of the surface is given by

$$T\left(\frac{1}{\rho} + \frac{1}{\rho'}\right),$$

where ρ and ρ′ are the principal radii of curvature of the surface.

In the case of two-dimensional waves we have $\rho' = \infty$ and, if η denote the elevation,

$$\frac{1}{\rho} = -\frac{d^2\eta}{dx^2},$$

neglecting squares of small quantities. So if δp, δp′ denote the variable parts of the pressure below and above the surface, we have

$$T\frac{d^2\eta}{dx^2} + \delta p - \delta p' = 0 \tag{93}$$

as the surface condition.

4.11.1 Capillary waves in a channel of uniform depth

Let us use the method of **Section-4.9,** reducing the problem to one of steady motion by superposing a velocity $-c$ on the whole mass, where c is the velocity of propagation. We have

$$\psi = cy - A \sinh m(y + h) \sin mx, \tag{94}$$

and for the free surface

$$\eta = a \sin mx, \tag{95}$$

provided

$$ca - A \sinh mh = 0. \tag{96}$$

Using these in the Bernoulli's equation, the variable part of the pressure is given by

$$\frac{\delta p}{\rho} + ga \sin mx + \frac{1}{2} c^2 \; (1 - 2\, am \coth mh \sin mx) = \text{constant}, \tag{97}$$

where the terms containing a^2 have been neglected. Now if we suppose that pressure on the upper side of the interface is constant, then $\delta p' = 0$ in (93) and so (93) reduces to

$$\delta p = -T \frac{d^2 \eta}{dx^2}$$

$$= T\, am^2 \sin mx. \tag{98}$$

Substituting this value in the last equation and equating to zero the coefficient of sin mx, we get

$$c^2 = \left(\frac{g}{m} + \frac{Tm}{\rho} \right) \tanh mh. \tag{99}$$

When h is large compared to the wave length this becomes

$$c^2 = \frac{g}{m} + \frac{Tm}{\rho}. \tag{100}$$

4.12 Illustrative Solved Examples

Example 1

When simple harmonic waves of length λ are propagated over the surface of deep water, prove that, at a point whose depth below the undisturbed surface is h, the pressure at the instants when the disturbed depth of the point is h + η bears to the undisturbed pressure at the same point the ratio

$$1+\frac{\eta}{h}e^{-2\pi h/\lambda} : 1,$$

atmospheric pressure and surface tension being neglected.

Solution :

For deep water, the velocity potential is given by

$$\phi = \frac{na}{m}e^{my}\cos(mx - nt), \qquad (1)$$

therefore

$$\frac{\partial\phi}{\partial t} = \frac{an^2}{m}e^{my}\sin(mx - nt). \qquad (2)$$

Also

$$\eta = a\sin(mx - nt)\ c^2 = \frac{n^2}{m^2} = \frac{g}{m}.$$

So (2) becomes

$$\frac{\partial\phi}{\partial t} = g\eta e^{my}. \qquad (3)$$

Pressure at any point within the water is given by

$$\frac{p}{\rho} - \frac{\partial\phi}{\partial t} + gy = C \text{ (a constant)}. \qquad (4)$$

When $y = 0$, $p = 0$, $\frac{\partial\phi}{\partial t} = 0$ so

$$C = 0$$

and hence (4) gives

$$p = \rho \frac{\partial \phi}{\partial t} - g\rho y$$

or,

$$p = g\rho\eta e^{my} - g\rho y, \quad \text{by (3)}, \tag{5}$$

Therefore disturbed pressure p_1 when $y = -h$ is given by

$$p_1 = \rho g\eta e^{-mh} + \rho gh$$

$$= \rho gh\left(1 + \frac{\eta}{h} e^{-mh}\right) \tag{6}$$

and undisturbed pressure p_2 at a depth h is given by

$$p_2 = \rho gh. \tag{7}$$

Therefore

$$p_1 : p_2 = \left(1 + \frac{\eta}{h} e^{-mh}\right) : 1$$

$$= \left(1 + \frac{\eta}{h} e^{-2\pi h/\lambda}\right) : 1, \quad (\text{since } m = 2\pi/\lambda).$$

Example 2

Shew that, if the velocity of the wind is just great enough to prevent the propagation of waves of length λ against it, the velocity of propagation of waves with the wind is

$$2\,c\left\{\frac{\sigma}{(1+\sigma)}\right\}^{\frac{1}{2}},$$

where σ is the specific gravity of air and c is the wave velocity when no air is present.

Solution :

If V, V′ be the velocities of the lower and upper of two liquids of densities ρ, ρ' and depths h, h′, then

$$g(\rho - \rho') = m[(V - c_1)^2 \rho \coth mh + (V' - c_1)^2 \rho' \coth mh']. \tag{1}$$

Given $\frac{\rho'}{\rho} = \sigma$. Since the sea is at rest, V = 0 and h and h′ both $\to \infty$. Hence (1) reduces to

$$g(1-\sigma) = m\left[c_1^2 + (V'-c_1)^2\,\sigma\right]. \quad (2)$$

If no wind is present, V′ = 0, then

$$c_1 = c.$$

Therefore from (2),

$$g(1-\sigma) = m(c^2 + c^2\sigma)$$
$$= mc^2(1+\sigma). \quad (3)$$

When there is no wave, $c_1 = 0$. From (2),

$$g(1-\sigma) = mV'^2\sigma. \quad (4)$$

Now from (2),

$$g(1-\sigma) = m\left(c_1^2 + V'^2\sigma + c_1^2\sigma - 2V'c_1\sigma\right)$$

or,

$$mV'^2\sigma = m\left(c_1^2 + V'^2\sigma + c_1^2\sigma - 2V'c_1\sigma\right), \quad \text{using (4)}$$

or,

$$c_1^2\,(1+\sigma) - 2V'c_1\sigma = 0$$

$$V' = \frac{c_1\,(1+\sigma)}{2\sigma}. \quad (5)$$

Putting this value of V′ in (4), we get

$$g(1-\sigma) = m\frac{c_1^2\,(1+\sigma)^2}{4\sigma^2}$$

or,

$$mc^2\,(1+\sigma) = m\sigma\frac{c_1^2\,(1+\sigma)^2}{4\sigma^2}$$

or,

$$c_1^2 = \frac{4\sigma}{1+\sigma} c^2,$$

so that

$$c_1 = 2c\left(\frac{\sigma}{1+\sigma}\right)^{\frac{1}{2}}.$$

Example 3

If there be two liquids in a straight channel of uniform section, of densities ρ_1, ρ_2 and depths l_1, l_2, shew that the velocity c of propagation of long waves is given by the equation

$$\left(\frac{c^2}{l_1 g} - 1\right)\left(\frac{c^2}{l_2 g} - 1\right) = \frac{\rho_1}{\rho_2},$$

where $\rho_2 > \rho_1$, and it is assume that the liquids do not mix.

Solution :

Proceeding as in **4.9.1** with $\rho = \rho_2$, $\rho' = \rho_1$, $h = l_2$, $h' = l_1$, we get from (25), of **4.9.1**,

$$c^4 m^2(\rho_2 \coth ml_2 \coth ml_1 + \rho_1) - c^2 gm\rho_2 (\coth ml_2 + \coth ml_1) + g^2(\rho_2 - \rho_1) = 0.$$

But for long waves, m is small and so we have

$$\coth ml_1 = \frac{1}{ml_1}$$

and

$$\coth ml_2 = \frac{1}{ml_2}$$

approximately. Therefore

$$\frac{c^4 \rho_2}{l_1 l_2} + \rho_1 m^2 c^4 - c^2 g\rho_2 \left(\frac{1}{l_1} + \frac{1}{l_2}\right) + g^2 (\rho_2 - \rho_1) = 0.$$

But for long waves,

$$m = \frac{2\pi}{\lambda}$$

is small. So neglecting m^2, we get

$$\frac{c^4}{l_1 l_2 g^2} - \frac{c^2}{g}\left(\frac{1}{l_1} + \frac{1}{l_2}\right) + 1 = \frac{\rho_1}{\rho_2}$$

or,

$$\left(\frac{c^2}{l_1 g} - 1\right)\left(\frac{c^2}{l_2 g} - 1\right) = \frac{\rho_1}{\rho_2}.$$

Example 4

Prove that

$$w = A\cos\frac{2\pi}{\lambda}(z + ih - ct)$$

is the complex potential for the propagation of simple harmonic surface waves of small high on water of depth h, the origin being in the undisturbed free surface. Express A in terms of the surface oscillations.

Solution :

We have for the progressive waves on the surface of water

$$\phi = \frac{ag}{n}\frac{\cosh m(y+h)}{\cosh mh}\cos(mx - nt). \tag{1}$$

Since

$$\cos m(x + iy) = \cos mx \cosh my - i \sin mx \sinh my,$$

we take

$$\psi = \frac{ag}{n}\frac{\sinh m(y+h)}{\cosh mh}\sin(mx - nt). \tag{2}$$

Therefore

$$w = \phi + i\psi$$

$$= \frac{ag}{n}\frac{\cos((mx - nt) + im(y+h))}{\cosh mh}$$

$$= \frac{ag}{n}\frac{\cos(m(x+iy) + imh - nt)}{\cosh mh}$$

$$= \frac{ag}{n} \frac{\cos m\left(z + ih - \frac{nt}{m}\right)}{\cosh mh}$$

$$= \frac{ag}{n} \frac{\cos \frac{2\pi}{\lambda}(z + ih - ct)}{\cosh mh}.$$

Since

$$m = \frac{2\pi}{\lambda}, c = \frac{n}{m}.$$

Therefore

$$w = A \cos \frac{2\pi}{\lambda}(z + ih - ct)$$

where

$$A = \frac{ag}{n \cosh mh} = \frac{ag}{mc \cosh mh}.$$

4.13 Model Questions

Short Questions :

1. Justify, by examples, the statement 'waves are means of propagation of energyy without any conspicuous movement of particles'.
2. What is meant by wave profile? Find the equation of the wave profile at any instant of time referred to a given origin.
3. Define : Simple harmonic progressive wave, standing (stationary) wave, surface wave, group velocity, capillary wave, long wave.
4. Show that a progressive wave can be regarded as due to the superposition of two standing waves.
5. Deduce the surface condition for capillary wave.

Broad Questions :

1. Deduce the condition at the free surface of an unbounded sheet of liquid for two-dimensional irrotational motion. Hence obtain the same if the motion be small.

Also show that in a wave motion in which the square of the velocities can be neglected, the velocity potential satisfies Laplace's equation and its normal derivative vanishes at a fixed boundary.

2. Discuss the motion of progressive waves (i) on the surface of water (ii) in deep water. Hence find the path of the particles in each case.

3. Deduce the expressions for the kinetic and potential energies of the progressive wave.

4. Discuss the motion of stationary waves (i) on the surface of water, (ii) in deep water. Hence find the path of the particles in each case.

5. Find the expression for the group velocity for waves on the surface of liquid of finite depth. Hence show that the group velocity for shallow water is equal to the wave velocity but that for deep sea waves is half the wave velocity.

6. Show that in a simple harmonic train of surface waves, energy crosses a fixed vertical plane perpendicular to the direction of propagation at an average rate equal to group velocity.

7. Find the rate of transmission of energy in simple harmonic surface waves.

8. Consider waves propagating in one direction without change of shape. Show how the problems of propagation of surface waves (i) on the surface of water and (ii) in a deep water, can be reduced to the problems of steady motion.

9. Discuss the motion of oscillatory waves at the interface of two liquids.

10. Discuss the motion of waves at the interface of two liquids with free upper surface.

11. What is meant by long wave? Show that for such waves, the points which lie once in a vertical plane always remain there.

12. Deduce the equations of motion and continuity for long waves. Hence find the solution of the equation of motion and interpret the result. Analyse the results for a canal of rectangular cross-section of given depth.

13. Discuss the motion of capillary waves in a channel of uniform depth.

Problems :

1. Let a shallow trough be filled with oil and water, and let the depth of the water be k and its density ρ_1, and the depth of the oil h and its density ρ_2. Then shew that if g be gravity, and v the velocity of propagation of long waves.

$$\frac{v^2}{g} = \frac{1}{2}(h+k) + \frac{1}{2}\left\{(h-k)^2 + \frac{4hk\rho_2}{\rho_1}\right\}^{\frac{1}{2}}.$$

Note that there may be slipping between the two fluids.

2. Two fluids of densities ρ_1, ρ_2 have a horizontal surface of separation but are otherwise unbounded. Shew that when waves of small amplitude are propagated at their common surface, the particles of the two fluids describe circles about their mean positions; and that at any point of the surface of separation where the elevation is η, the particles on either side have a relative velocity $\frac{4\pi c\eta}{\lambda}$.

3. If a channel of rectangular section contain a depth h of liquid of density ρ on which is superposed a depth h′ of liquid of density ρ', the free surface of the latter being exposed to constant atmospheric pressure, prove that the velocities of propagation of waves of length $2\pi/m$ are given by $c^2 = \frac{gu}{m}$, where

$$\rho(u \coth mh - 1)(u \coth mh' - 1) = \rho'(1 - u^2).$$

4. Two-dimensional waves of length $2\pi/m$ are produced at the surface of separation of two liquids which are of densities ρ, $\rho'(\rho > \rho')$ and depths h, h′ confined between two fixed horizontal planes. Prove that, if the potential energy is reckoned zero in the position of equilibrium, the total energy of the lower liquid is to that of the upper in the ratio

$$\rho((2\rho - \rho')\coth mh + \rho'\coth mh') : \rho'((\rho - 2\rho')\coth mh' - \rho\coth mh).$$

5. A channel, of infinite length and rectangular section, is of uniform depth h and breadth b in one part but changes gradually to uniform depth h′ and breath b′ in another part. An infinite train of simple harmonic waves travelling in one direction

only is propagated along the channel. Prove that, if a, a′ are the heights and 2π/m, 2π/m′ the lengths of the waves in the two uniform portions,

$$m \tanh mh = m' \tanh mh',$$

and

$$\frac{a^2 b}{\cosh^2 mh}(\sinh 2mh + 2mh) = \frac{a'^2 b'}{\cosh^2 m'h'}(\sinh 2m'h' + 2m'h').$$

4.14 Summary

The conception of surface waves relating to progressive and standing waves has been introduced. A sketch of long waves and capillary waves are also noted.

Unit 5 □ Viscous Flow

Structure

5.0 Introduction

5.1 Viscous incompressible flow : Navier-Stokes' equations

- **5.1.1 Flow through tube of uniform cross section**
- **5.1.2 Flow through a pipe of circular cross section**
- **5.1.3 Flow through pipe with annular cross section**
- **5.1.4 Flow through a pipe with elliptic cross section**
- **5.1.5 Flow through a pipe with rectangular cross section**

5.2 Boundary Layer

- **5.2.1 Concept of boundary layer**
- **5.2.2 Two dimensional boundary layer flow over a plane wall**
- **5.2.3 Boundary layer over a flat plate : (Blasius Solution)**
- **5.2.4 Shearing stress on the plate**
- **5.2.5 Boundary layer thickness**

5.3 Moderl Questions

5.4 Summary

5.0 Introduction

So far we have considered the motion of an ideal or non-viscous fluid, that is the fluid which is incapable of exerting shearing (i.e. tangential) stress on any surface with which it is in contact. We now proceed to introduce the fluid motion for which the normal and the shearing stresses will be taken into account. The resulting equations, known as Navier-Stokes' equations are of fundamental importance and what else follows will be based on these equations.

5.1 Viscous Incompressible Flow : Navier-Stokes' Equations

It has already been seen (see study Material PG(MT)05 : Group-B, Page-122) that for incompressible viscous fluid, Navier-Stokes equation of motion is given by

$$\frac{\partial \vec{v}}{\partial t}+\left(\vec{v}.\vec{\nabla}\right)\vec{v}=\vec{F}-\frac{1}{\rho}\vec{\nabla}p+\nu\nabla^2\vec{v} \tag{1}$$

where $\vec{v}$ is the velocity vector; $\vec{F}$, the external force; ρ, the fluid density; p, the pressure and ν is the kinematic coefficient of viscosity.

Let us now consider some deductions from the equation (1).

Vorticity transport equation

We rewrite the equation (1) in the form

$$\frac{\partial \vec{v}}{\partial t}+\vec{\nabla}\left(\frac{1}{2}v^2\right)+\vec{w}\times\vec{v}=\vec{F}-\frac{1}{\rho}\vec{\nabla}p+\nu\nabla^2\vec{v}$$

where $\vec{w}=\vec{\nabla}\times\vec{v}$ represents the vorticity vector. Assuming conservative nature of external forces so that $\vec{F}=-\vec{\nabla}\chi$, χ being potential function, we have from the above equation

$$\frac{\partial \vec{v}}{\partial t}+\vec{w}\times\vec{v}=-\vec{\nabla}\left(\chi+\frac{1}{2}v^2+\frac{p}{\rho}\right)+\nu\nabla^2\vec{v}.$$

Taking curl of both sides, it follows that

$$\frac{\partial \vec{w}}{\partial t}+\vec{\nabla}\times(\vec{w}\times\vec{v})=\nu\nabla^2\vec{w} \tag{2}$$

Now $\vec{\nabla}\times(\vec{w}\times\vec{v})=(\vec{\nabla}.\vec{v})\vec{w}-(\vec{\nabla}.\vec{w})\vec{v}+(\vec{v}.\vec{\nabla})\vec{w}-(\vec{w}.\vec{\nabla})\vec{v}$

$=(\vec{v}.\vec{\nabla})\vec{w}-(\vec{w}.\vec{\nabla})\vec{v}$ (Using equation of continuity $\vec{\nabla}.\vec{v}=0$ and the result $\vec{\nabla}.\vec{w}=\vec{\nabla}.\vec{\nabla}\times\vec{v}=0$)

so that equation (2) reduces to

$$\frac{\partial \vec{w}}{\partial t}+(\vec{v}.\vec{\nabla})\vec{w}-(\vec{w}.\vec{\nabla})\vec{v}=\nu\nabla^2\vec{w}$$

i.e., $$\frac{\partial \vec{w}}{\partial t}+(\vec{v}.\vec{\nabla})\vec{w}=(\vec{w}.\vec{\nabla})\vec{v}+\nu\nabla^2\vec{w} \tag{3a}$$

i.e., $$\frac{d\vec{w}}{dt}=(\vec{w}.\vec{\nabla})\vec{v}+\nu\nabla^2\vec{w} \tag{3b}$$

where $\frac{d}{dt} = \frac{\partial}{\partial t} + \vec{v}.\vec{\nabla}$. Equation (3a) or (3b) is known as *vorticity equation.*

Dissipation of energy

We now calculate the energy which is dissipated in a viscous liquid in motion due to internal friction.

Suppose the liquid is contained within a volume V bounded by a closed surface S. The forces acting on the liquid are the external force $\vec{F}$ per unit mass, the normal pressure p on the boundary and the viscous stress acting over the surface S. Now the rate at which the work is done by these forces is

$$\int_V \rho F_i v_i \, dv + \int_S (T_{ij} n_j) v_i \, ds = \int_V \left[\rho F_i v_i + \frac{\partial}{\partial x_j}(T_{ij} v_i) \right] dv \qquad (4)$$

where T_{ij} is the stress given by

$$T_{ij} = -p\delta_{ij} + \mu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_i}{\partial x_i} \right),$$

(see study Material PG(MT)05 : Group-B, page-122)

μ being the viscosity.

Let K be the kinetic energy and E be the intrinsic energy so that

$$K + E = \frac{1}{2}\int_V \rho \vec{v}.\vec{v} \, dv + \int_V \rho e \, dv,$$

e being the intrinsic energy per unit mass. Then the rate of increase of this total energy is

$$\frac{d}{dt}(K + E) = \int_V \rho \vec{v}.\frac{d\vec{v}}{dt} dv + \int_V \rho \frac{de}{dt} dv = \int_V \rho\left(v_i \frac{dv_i}{dt} + \frac{de}{dt} \right) dv \qquad (5)$$

so that from (4) and (5) we get by using the principle of energy

$$\int_V \rho\left(\frac{de}{dt} + v_i \frac{dv_i}{dt} \right) dv = \int_V \left[\rho F_i v_i + \frac{\partial}{\partial x_j}(T_{ij} v_i) \right] dv$$

$$\text{or,} \qquad \int_V \left\{ \rho \frac{de}{dt} + \rho v_i \frac{dv_i}{dt} - \rho F_i v_i - \frac{\partial}{\partial x_j}(T_{ij} v_i) \right\} dv = 0.$$

Since this is true for arbitrary volume V, we must have

$$\rho \frac{de}{dt} = \rho F_i v_i - \rho v_i \frac{dv_i}{dt} + \frac{\partial}{\partial x_j}(T_{ij} v_i) \qquad (6)$$

Noting that

$$\frac{\partial}{\partial x_j}(T_{ij} v_i) = T_{ij} \frac{\partial v_i}{\partial x_j} + v_i \frac{\partial T_{ij}}{\partial x_j}$$

$$= T_{ij} (e_{ij} + w_{ij}) + v_i \left(\frac{dv_i}{dt} - F_i \right) \rho$$

(see equation (4.16) in study Material PG(MT)05 : Group-B, page 66)

where $\frac{d}{dt} = \frac{\partial}{\partial t} + \vec{v} . \vec{\nabla}$, $e_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right)$ and $w_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i} \right)$ and since T_{ij} are symmetric but w_{ij} are skew-symenetric so that $T_{ij}w_{ij} = 0$, we have

$$\frac{\partial}{\partial x_j}(T_{ij} v_i) = T_{ij} e_{ij} + \rho v_i \left(\frac{dv_i}{dt} - F_i \right)$$

Thus from (5), we get

$$\rho \frac{de}{dt} = \rho F_i v_i - \rho v_i \frac{dv_i}{dt} + T_{ij} e_{ij} + \rho v_i \left(\frac{dv_i}{dt} - F_i \right)$$

i.e. $$\rho \frac{de}{dt} = T_{ij} e_{ij} = \left[-p\delta_{ij} + \mu \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \right] e_{ij}$$

$$= -pe_{ii} + \mu \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) e_{ij}$$

$$= -p \frac{\partial v_i}{\partial x_i} + \frac{1}{2}\mu \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right)^2$$

$$\therefore \quad \rho \frac{de}{dt} = -p \frac{\partial v_i}{\partial x_i} + \Phi$$

where $\Phi = \frac{1}{2}\mu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)^2$ is the dissipation function and is necessarily positive. In Cartesian form

$$\Phi = \frac{1}{2}\mu\left\{\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)^2\right\}$$

Obviously, this expression is never negative and is zero only when each of the squared term vanishes. It is, therefore, evident that energy is always dissipated and reappears in the form of heat unless liquid moves without any strain, that is as a rigid body.

We now proceed to discuss the steady motion of incompressible viscous liquids through different tubes and channel.

5.1.1 Flow through tube of uniform cross-section

We consider steady flow of imcompressible viscous flow through a tube of arbitrary but uniform cross-section. We take the z-axis along the axis of the pipe. We suppose that only the non-zero velocity component is along the z-axis, so we put $u = 0$, $v = 0$, $w \neq 0$.

Under this assumption the set of basic equations are

$$\frac{\partial w}{\partial z} = 0 \quad \text{(equation of continuty)}, \tag{7}$$

$$\frac{\partial p}{\partial x} = 0 \quad \text{(equation of motion along x-direction)}, \tag{8}$$

$$\frac{\partial p}{\partial y} = 0 \quad \text{(equation of motion along y-direction)}, \tag{9}$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) \text{(equation of motion along z-direction)}. \tag{10}$$

(7) implies that w is a function of x and y only and is independent of z. (8) and (9) imply that p is a function of z only. Thus (10) becomes

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \frac{1}{\mu}\frac{dp}{dz}. \tag{11}$$

L.H.S. of (11) is a function of x and y whereas R.H.S. of (11) is a function of z only. So each must be constant. We write

$$\frac{dp}{dz} = -P. \tag{12}$$

We have considered the negative sign because we expect that pressure P decreases in the direction of flow. So the equation satisfied by w is

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = -\frac{P}{\mu}. \tag{13}$$

This equation is to be solved subject to the boundary condition w = 0 on the surface of the tube. We simplify the equation by writing

$$w = \psi - \frac{P}{4\mu}(x^2 + y^2) \tag{14}$$

so that

$$\frac{\partial w}{\partial x} = \frac{\partial \psi}{\partial x} - \frac{Px}{2\mu}, \frac{\partial^2 w}{\partial x^2} = -\frac{P}{2\mu} + \frac{\partial^2 \psi}{\partial x^2},$$

$$\frac{\partial w}{\partial y} = \frac{\partial \psi}{\partial y} - \frac{Py}{2\mu}, \frac{\partial^2 w}{\partial y^2} = -\frac{P}{2\mu} + \frac{\partial^2 \psi}{\partial y^2}.$$

Hence,

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = -\frac{P}{2\mu} + \frac{\partial^2 \psi}{\partial x^2} - \frac{P}{2\mu} + \frac{\partial^2 \psi}{\partial y^2} = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} - \frac{P}{\mu}.$$

Since

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = -\frac{P}{\mu}$$

we obtain

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0. \tag{15}$$

Therefore, ψ satisfies the two-dimensional Laplace equation with the boundary condition

$$\psi = \frac{P}{4\mu}(x^2 + y^2) \tag{16}$$

on the surface of the tube.

5.1.2 Flow through a pipe of circular cross-section

The equation of the cross-section of the pipe is $x^2 + y^2 = a^2$, or $r = a$. Boundary condition is $\psi = \frac{P}{4\mu}a^2$ on $r = a$. To satisfy this condition we choose $\psi = A = \frac{Pa^2}{4\mu} =$ constant. Therefore the velocity w is given by,

$$w = \psi - \frac{P}{4\mu}(x^2 + y^2) = \frac{Pa^2}{4\mu} - \frac{Pr^2}{4\mu} = \frac{P}{4\pi}(a^2 - r^2).$$

Hence

$$w(r) = \frac{P}{4\mu}(a^2 - r^2). \tag{17}$$

The form (15) shows that the velocity profile is parabolic, i.e., the plot of w against r from $r = 0$ to $r = a$ is of parabolic shape. The volume rate of flow at any cross section is given by,

$$Q = \int_0^a w(r).2\pi r dr = \int_0^a \frac{P}{4\mu}(a^2 - r^2)2\pi r dr = \frac{P}{4\mu}2\pi\int_0^a (a^2 - r^2) r dr$$

$$= \frac{P\pi}{2\mu}\int_0^a (a^2 r - r^3) dr = \frac{P\pi}{2\mu}\left[a^2 . \frac{a^2}{2} - \frac{a^4}{4}\right] = \frac{P\pi a^4}{8\mu}.$$

It is clear that the pressure gradient $\frac{dp}{dz} = \frac{p_2 - p_1}{l}$, where p_1 and p_2 are the pressures at two sections at a distance l apart. So the volume rate of flow.

$$Q = \frac{\pi a^4}{8\mu}P = \frac{\pi a^4}{8\mu}\left(-\frac{dp}{dz}\right) = \frac{\pi a^4}{8\mu}(p_1 - p_2). \tag{18}$$

This formula is used to determine the coefficient of viscosity μ. Since all other quantities can be measured experimentally, μ can be determined from the formula (16).

5.1.3 Flow through a pipe with annular cross-section

Consider the pipe $b < r < a$, i.e., the region between two concentric cylinders $r = b$ and $r = a$. The boundary conditions are $\psi = \frac{Pb^2}{4\mu}$ on the inner cylinder $r = b$ and $\psi = \frac{Pa^2}{4\mu}$ on the outer cylinder $r = a$.

An appropriate choice of ψ satisfying the Laplace equation in $a < r < b$ is $\psi = A + B \ln r$. By the boundary conditions, we find

$$\frac{Pb^2}{4\mu} = A + B\ln b, \quad \frac{Pa^2}{4\mu} = A + B\ln a.$$

Thus

$$B = \frac{P}{4\mu}\left(\frac{b^2 - a^2}{\ln b - \ln a}\right) = \frac{P}{4\mu}\frac{(b^2 - a^2)}{\ln\left(\frac{b}{a}\right)},$$

and

$$A = \frac{Pa^2}{4\mu} - B\ln a = \frac{Pa^2}{4\mu} - \left[\frac{P}{4\mu}\frac{(b^2 - a^2)}{\ln\left(\frac{b}{a}\right)}\right]\ln a.$$

Hence

$$\psi = \frac{Pa^2}{4\mu} - \left[\frac{P}{4\mu}\frac{(b^2 - a^2)}{\ln\left(\frac{b}{a}\right)}\right] + B\ln r = \frac{Pa^2}{4\mu}\left[\frac{P}{4\mu}\frac{(b^2 - a^2)}{\ln\left(\frac{b}{a}\right)}\right]\ln\left(\frac{r}{a}\right)$$

so that

$$w = \psi - \frac{P}{4\mu}(x^2 + y^2) = \psi - \frac{P}{4\mu}r^2.$$

Hence

$$w = \frac{P}{4\mu}\left[(a^2 - r^2) + (b^2 - a^2)\frac{\ln(b/a)}{\ln(r/a)}\right]. \tag{19}$$

The rate of volume flow is given by

$$Q = \int_b^a w(r).2\pi r dr = \frac{P\pi}{8\mu}\left[a^4 - b^4 - \frac{(a^2 - b^2)^2}{\ln\left(\frac{a}{b}\right)}\right]$$

5.1.4 Flow through a pipe with elliptic cross-section

Let the equation of cross section of the pipe be,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1. \qquad (20)$$

A suitable solution for the Laplace equation for this case is

$$\psi = A(x^2 - y^2) + B. \qquad (21)$$

To satisfy the boundary condition w = 0 on the surface of the pipe,

$$\psi = \frac{P}{4\mu}(x^2 + y^2) = A(x^2 - y^2) + B$$

on the surface of the pipe. This implies that

$$\left(\frac{P}{4\mu} - A\right)x^2 + \left(\frac{P}{4\mu} + A\right)y^2 = B$$

$$\text{i.e., } \frac{x^2}{\dfrac{B}{\frac{P}{4\mu}-A}} + \frac{y^2}{\dfrac{B}{\frac{P}{4\mu}+A}} = 1.$$

Comparing (18), (20) we obtain,

$$\frac{B}{\frac{P}{4\mu} - A} = a^2 \; ; \; \frac{B}{\frac{P}{4\mu} + A} = b^2$$

$$\Rightarrow \frac{P}{4\mu} - A = \frac{B}{a^2}; \; \frac{P}{4\mu} + A = \frac{B}{b^2}$$

$$\Rightarrow \frac{P}{2\mu} = B\left(\frac{1}{a^2} + \frac{1}{b^2}\right) = B\left(\frac{a^2 + b^2}{a^2 b^2}\right)$$

$$\Rightarrow B = \frac{Pa^2 b^2}{2\mu(a^2 + b^2)}.$$

Hence,

$$A = \frac{P}{4\mu} - \frac{B}{a^2} = \frac{P}{4\mu} - \frac{Pb^2}{2\mu(a^2 + b^2)} = \frac{P}{4\mu}\left[1 - \frac{2b^2}{a^2 + b^2}\right] = \frac{P}{4\mu}\left(\frac{a^2 - b^2}{a^2 + b^2}\right).$$

Hence the velocity distribution w is given by

$$w = \psi - \frac{P}{4\mu}(x^2 + y^2) = \frac{P}{4\mu}\left(\frac{a^2 - b^2}{a^2 + b^2}\right)(x^2 - y^2) + \frac{Pa^2 b^2}{2\mu(a^2 + b^2)} - \frac{P}{4\mu}(x^2 + y^2)$$

$$= \frac{Pa^2 b^2}{2\mu(a^2 + b^2)}\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right).$$

The rate of volume flow is give by

$$M = \iint w dxdy = \frac{Pa^2 b^2}{2\mu(a^2 + b^2)}\iint\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)dxdy.$$

Now we consider the ellipse $x = a\lambda \cos \lambda$, $y = b\lambda \sin \lambda$ i.e., $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \lambda^2$. On this ellipse, the integrand $\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right) = 1 - \lambda^2 (\sin^2 \lambda + \cos^2 \lambda) = 1 - \lambda^2$.

Now the area between this ellipse and the neighbouring ellipse (where λ is increased by $\lambda + d\lambda$) is

$= \pi a(\lambda + d\lambda)b(\lambda + d\lambda) - \pi a\lambda b\lambda = \pi ab(\lambda + d\lambda)^2 - \pi ab\lambda^2 = 2\pi ab\lambda d\lambda$.

Therefore

$$M = \frac{Pa^2 b^2}{2\mu(a^2 + b^2)}\int_0^1 (1 - \lambda^2).2\pi ab\lambda d\lambda = \frac{\pi P}{\mu}.\frac{a^3 b^3}{a^2 + b^2}\int_0^1 \lambda(1 - \lambda^2)d\lambda$$

$$= \frac{\pi P}{4\mu}\frac{a^3 b^3}{a^2 + b^2}.$$

Now the rate of volume flow through a pipe of circular cross-section with radius $(ab)^{1/2}$ having the same cross section as the ellipse is $M_c = \frac{\pi P}{8\mu}a^2 b^2$.

$$\frac{M}{M_c} = \frac{\pi P}{4\mu}\cdot\frac{a^3 b^3}{a^2 + b^2} \times \frac{8\mu}{\pi P}\cdot\frac{1}{a^2 b^2} = \frac{2ab}{a^2 + b^2} < 1$$

$$\Rightarrow M < M_c$$

Thus the flux through a circle is greater than that through an ellipse. The physical reason is that for a given pressure gradient the rate of flow is diminished by the friction. Now this friction is minimum on a circle because among all curves with the same enclosed area circle is the curve of minimum periphery.

5.1.5 Flow through a pipe with rectangular cross-section

Let the cross section be bounded by the planes $x = a$, $x = -a$ and $y = b$, $y = -b$. We have to solve

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = -\frac{P}{\mu} \tag{22}$$

subject to the boundary conditions,

$$\text{(i) } w = 0 \text{ at } x = a,\ x = -a,$$

$$\text{(ii) } w = 0 \text{ at } y = b,\ y = -b. \tag{23}$$

One particular solution of (22) satisfying the boundary condition is

$$w_1 = \frac{P}{2\mu}(a^2 - x^2). \tag{24}$$

If we write, $w = w_1 + w_2$ then,

$$\frac{\partial^2 w_2}{\partial x^2} + \frac{\partial^2 w_2}{\partial y^2} = 0. \tag{25}$$

We solve this equation (25) by method of separation of variables where we assume

$$w_2(x, y) = X(x)Y(y). \tag{26}$$

Substituting (26) in (25) we get,

$$\frac{d^2X(x)}{dx^2}Y(y) + X(x)\frac{d^2Y(y)}{dy^2} = 0$$

$$\Rightarrow -\frac{1}{X(x)}\frac{d^2X(x)}{dx^2} = \frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2} = C_n^2$$

$$\Rightarrow \frac{d^2X(x)}{dx^2} = -C_n^2 X(x);\ \frac{d^2Y(y)}{dy^2} = -C_n^2 Y(y).$$

Solutions are

$$X(x) = A\cos(C_n x) + B\sin(C_n x),\quad Y(y) = A\cos(C_n y) + B\sin(C_n y).$$

Now, from the symmetry of the cross section with respect to both x and y, it follows that w must be even function of x and y. Since, from (3) w_1 is already even in x, w_2 must be even in x, w_2 must be even in x and y. Hence

$$B = D = 0,$$

and therefore

$$w_2 = \sum_{n=0}^{\infty} A_n \cos(C_n x)\cosh(C_n y).$$

Here,

$$w = w_1 + w_2 = \frac{P}{4\mu}(a^2 - x^2) + \sum_{n=0}^{\infty} A_n \cos(C_n x)\cosh(C_n y).$$

To satisfy the boundary condition w = 0 at x = a, x = – a, we have

$$0 + \sum_{n=0}^{\infty} A_n \cos(C_n a)\cosh(C_n y)$$

$$\Rightarrow \cos(C_n a) = 0, \text{ i.e., } C_n a = (2n+1)\frac{\pi}{2}, \text{ i.e., } C_n = (2n+1)\frac{\pi}{2a}$$

Therefore

$$w = \frac{P}{2\mu}(a^2 - x^2) + \sum_{n=0}^{\infty} A_n \cos\left\{(2n+1)\frac{\pi x}{2a}\right\}\cosh\left\{(2n+1)\frac{\pi y}{2a}\right\},$$

By the boundary condition (ii) w = 0 at y = b, y = – b and we have,

$$0 = \frac{P}{2\mu}(a^2 - x^2) + \sum_{n=0}^{\infty} A_n \cos\left\{(2n+1)\frac{\pi x}{2a}\right\}\cosh\left\{(2n+1)\frac{\pi b}{2a}\right\}$$

$$\Rightarrow -\frac{P}{2\mu}(a^2 - x^2) = \sum_{n=0}^{\infty} A_n \cos\left\{(2n+1)\frac{\pi x}{2a}\right\}\cosh\left\{(2n+1)\frac{\pi b}{2a}\right\}.$$

Multiplying both sides by $\cos\left\{(2n+1)\frac{\pi x}{2a}\right\}$ and integrating between –a and a, we get,

$$-\frac{P}{2\mu}\int_{-a}^{a}(a^2-x^2)\cos\left\{(2n+1)\frac{\pi x}{2a}\right\}dx$$

$$=A_n\cosh\left\{(2n+1)\frac{\pi b}{2a}\right\}\int_{-a}^{a}\cos^2\left\{(2n+1)\frac{\pi x}{2a}\right\}dx$$

$$\Rightarrow -\frac{P}{2\mu}\left[\frac{2a^3}{\pi(2n+1)}\sin\left\{(2n+1)\frac{\pi x}{2a}\right\}-\frac{2a}{\pi(2n+1)}x^2\sin\left\{(2n+1)\frac{\pi x}{2a}\right\}\right]_{-a}^{a}$$

$$-\frac{P}{2\mu}\left[\frac{4a}{\pi(2n+1)}\int_{-a}^{a}x\sin\left\{(2n+1)\frac{\pi x}{2a}\right\}dx\right]$$

$$=\frac{A_n}{2}\cosh\left\{(2n+1)\frac{\pi b}{2a}\right\}\int_{-a}^{a}\left\{1+\cos^2(2n+1)\frac{\pi x}{2a}\right\}dx$$

$$\Rightarrow -\frac{P}{2\mu}\left[-\frac{8a^2}{\pi^2(2n+1)}x\cos\left\{(2n+1)\frac{\pi x}{2a}\right\}+\frac{16a^3}{\pi^3(2n+1)^3}\sin\left\{(2n+1)\frac{\pi x}{2a}\right\}\right]_{-a}^{a}$$

$$=\frac{A_n}{2}\cosh\left\{(2n+1)\frac{\pi b}{2a}\right\}.2a$$

$$\Rightarrow -\frac{P}{2\mu}\frac{32a^3}{\pi^3(2n+1)^3}(-1)^n=aA_n\cosh\left\{(2n+1)\frac{\pi b}{2a}\right\}$$

$$\Rightarrow A_n=-\frac{P}{2\mu}\times\frac{32a^2(-1)^n}{\pi^3(2n+1)^3}\cosh\left\{(2n+1)\frac{\pi b}{2a}\right\}.$$

Hence

$$w=\frac{P}{2\mu}(a^2-x^2)-\frac{P}{\mu}\sum_{n=0}^{\infty}\frac{16a^2(-1)^n}{\pi^3(2n+1)^3}\cosh\left\{(2n+1)\frac{\pi b}{2a}\right\}$$

$$\times\cos\left\{(2n+1)\frac{\pi x}{2a}\right\}\cosh\left\{(2n+1)\frac{\pi y}{2a}\right\}.$$

5.2 Boundary Layer

5.2.1 Concept of boundary layer

The concept of boundary layer was introduced by Prandtl in 1904. He assumed that for fluid with small viscosity, the flow around a solid body can be divided into two parts

(i) a very thin layer called boundary, adjacent to the boundary layer where viscous effect is important and

(ii) a region outside the boundary where viscous effect is not important the flow may be taken as potential flow. Within this boundary layer, the Navier-Stokes equation can be simplified. These are called the boundary layer equations.

5.2.2 Two dimensional boundary layer flow over a plane wall

For motion in the (x, y)-plane, the Navier-Stokes equation and the equation of continuity are,

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \tag{27}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right), \tag{28}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{29}$$

Here x-axis is taken along the wall and y-axis normal to the wall. Due to no slip condition, $u = v = 0$ at the wall. Let $U(x, t)$ be the velocity outside the boundary layer. Then the velocity component u within the boundary layer rises rapidly from its value 0 at the wall to the value U at a small distance $\delta(x)$. δ is the boundary layer thickness and $\delta \ll 1$. We now calculate the order of magnitude of viscous terms in the equation of motion. We take u, x, t are of $O(1)$, but $y = O(\delta)$. By the equation of continuity,

$$\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} = O(1)$$

so that

$$v = O(\delta).$$

Then from (1),

$$\frac{\partial u}{\partial t} = O(1),\ \frac{\partial u}{\partial x} = O(1),\ v\frac{\partial v}{\partial y} = O(\delta).O\left(\frac{1}{\delta}\right) = O(1),$$

$$\frac{\partial^2 u}{\partial x^2} = O(1),\ \frac{\partial^2 u}{\partial y^2} = O\left(\frac{1}{\delta^2}\right).$$

Thus in equation (27) we can neglect the term $\frac{\partial^2 u}{\partial x^2}$ compared to $\frac{\partial^2 u}{\partial y^2}$. So equation (27) becomes

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + v\frac{\partial^2 u}{\partial y^2}. \tag{30}$$

If each term of this equation is of the same order of magnitude, we note that,

$$v = O(\delta^2),\ \Rightarrow \delta = O(\sqrt{v}).$$

Now we consider equation (28). We see that,

$$\frac{\partial v}{\partial t} = O(\delta),\ u\frac{\partial v}{\partial x} = O(\delta),\ v\frac{\partial v}{\partial y} = O(\delta),\ \frac{\partial^2 v}{\partial x^2} = O(\delta),\ \frac{\partial^2 v}{\partial y^2} = O\left(\frac{1}{\delta}\right)$$

Therefore,

$$-\frac{1}{\rho}\frac{\partial p}{\partial y} = O(\delta). \tag{31}$$

Thus the pressure gradient normal to the wall is of the order of δ. Hence integrating (31) with respect to y from $y = 0$ to δ, the pressure p may be neglected. Thus within the boundary layer pressure p may be taken as a function of x only and is given by its value at the outer edge of the boundary layer. Suppose that the flow outside the boundary layer is given by U(x, t). Then,

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} = \frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x}.$$

Thus Prandtl's boundary layer equations are,

$$\frac{\partial u}{\partial x} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + v\frac{\partial^2 U}{\partial y^2}, \tag{32a}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0. \tag{32b}$$

Boundary conditions are $u = v = 0$ at $y = 0$ and $u = U(x, t)$, at $y \to \infty$. The first boundary condition is the usual no slip condition.

It may be seen that considerable simplification has been achieved in the above equations which consist of two equations with two unknowns u and v. However the equations are still nonlinear, therefore it has been possible to solve the equations directly only for a limited number of problems, such as flow past a flat plate.

5.2.3 Boundary layer over a flat plate : (Blasius Solution)

The first application of Prandtl's boundary layer equations was made by H. Blasius (1908) to determine analytically an expression for thickness of the boundary layer over a wide semi infinite plate.

Now consider the steady flow of viscous incompressible fluid past a semi infinite plate placed in the direction of the uniform stream with velocity U.

We take the origin at the leading edge of the plate, x-axis along the plate and y-axis normal to the plate. In this case, the potential flow outside the boundary layer equations are,

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=v\frac{\partial^2 U}{\partial y^2}\quad\left[\text{since},\ \frac{\partial U}{\partial t}+U\frac{\partial U}{\partial x}\ \text{as}\ U=\text{constant}\right], \tag{33a}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{33b}$$

with the boundary condition $u = v = 0$ at $y = 0$ and $u = U$ at $y \to \infty$.

The equation of continuity can be integrated introducing by the stream function $\psi(x, y)$,

$$u=\frac{\partial \psi}{\partial y}\quad v=-\frac{\partial \psi}{\partial x}. \tag{34}$$

The characteristic parameters of this flow are U, x, y and v. i.e., the problem is determined in terms of these parameters. We may write

$$u = U\,F(\eta)\ \text{where}\ \eta=\left(\frac{U}{vx}\right)^{1/2} y\left[\text{Here}\ \delta=\sqrt{\frac{vx}{U}}\ \text{is the boundary layer thickness}\right].$$

Now by the first relation of (33), we get,

$$\psi = \int u dy = U \int F(\eta)\left(\frac{\nu x}{U}\right)^{1/2} d\eta = (\nu x U)^{1/2} \int F(\eta) d\eta$$

$$= (\nu x U)^{1/2} f(\eta) \qquad \left[\text{where, } f(\eta) = \int F(\eta) d\eta\right].$$

Now,

$$u = \frac{\partial \psi}{\partial y} = (\nu x U)^{1/2} \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial y} = (\nu x U)^{1/2}\left(\frac{U}{\nu x}\right)^{1/2} f'(\eta) = Uf'(\eta)$$

and

$$v = -\frac{\partial \psi}{\partial x} = -(\nu U)^{1/2} \frac{1}{2} x^{-1/2} f(\eta) - (\nu x U)^{1/2} \frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial x}$$

$$= -\frac{1}{2}\left(\frac{\nu U}{x}\right)^{1/2} f(\eta) - (\nu x U)^{1/2} f'(\eta)\left(-\frac{1}{2}\right)\left(\frac{U}{\nu}\right)^{1/2} \frac{y}{x^{3/2}}$$

$$= -\frac{1}{2}\left(\frac{\nu U}{x}\right)^{1/2} f(\eta) + \frac{1}{2}(\nu x U)^{1/2} f'(\eta).\frac{\eta}{x}.$$

Hence,

$$v = \frac{1}{2}\left(\frac{\nu U}{x}\right)^{-1/2}\left\{-f(\eta) + \eta f'(\eta)\right\},$$

$$\frac{\partial u}{\partial x} = Uf'(\eta)\frac{\partial \eta}{\partial x} = -\frac{1}{2}U\frac{\eta}{x} f''(\eta),$$

$$\frac{\partial u}{\partial y} = Uf''(\eta)\frac{\partial \eta}{\partial y} = Uf''(\eta)\left(\frac{U}{\nu x}\right)^{1/2} = U\left(\frac{U}{\nu x}\right)^{1/2} f''(\eta),$$

$$\frac{\partial^2 u}{\partial y^2} = U\left(\frac{U}{\nu x}\right)^{1/2} f'''(\eta)\frac{\partial \eta}{\partial y} = U\left(\frac{U}{\nu x}\right)^{1/2} f'''(\eta)\left(\frac{U}{\nu x}\right)^{1/2} = \frac{U^2}{\nu x} f'''(\eta).$$

Substituting all these terms into equation (1) we get,

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial y^2}$$

$$\Rightarrow Uf'(\eta)\left(-\frac{1}{2}\right)U\frac{\eta}{x}f''(\eta)+\frac{1}{2}\left(\frac{\nu U}{x}\right)^{1/2}$$

$$\{-f(\eta)+\eta f'(\eta)\}U\left(\frac{U}{\nu x}\right)^{1/2}f''(\eta)=\nu\frac{U^2}{\nu x}f'''(\eta)$$

$$\Rightarrow -\frac{1}{2}\frac{U^2}{x}\eta f'f''+\frac{1}{2}\frac{U^2}{x}f''(\eta f'-f)=\frac{U^2}{x}f'''$$

$$\Rightarrow 2f'''=-\eta f'f''+f'(\eta f'-f)\Rightarrow 2f'''+ff''=0$$

$$\Rightarrow 2\frac{d^3f}{d\eta^3}+f\frac{d^2f}{d\eta^2}=0. \tag{35}$$

So the boundary conditions are

(i) $f = f' = 0$ at $\eta = 0$ (i.e., $y = 0$),

(ii) $f' = 1$ as $y \to \infty$.

Since the equation (35) is a nonlinear equation, its solution in closed form is not possible. Blasius solved it by power series expansion of $f(\eta)$ about $\eta = 0$. We assume,

$$f(\eta)=A_0+A_1\eta+\frac{A_2}{2!}\eta^2+\frac{A_3}{3!}\eta^3+\cdots$$

By the boundary conditions

(i) $f = 0$ at $\eta = 0 \Rightarrow A_0 = 0$

(ii) $f' = 0$ at $\eta = 0 \Rightarrow A_1 = 0$.

Substituting these in the differential equation (35), we get,

$$f(\eta)=\frac{A_2}{2!}\eta^2+\frac{A_3}{3!}\eta^3+\cdots$$

$$f'(\eta)=A_2\eta+\frac{A_3}{2}\eta^2+\frac{A_4}{6}\eta^3+\frac{A_5}{24}\eta^4+\frac{A_6}{120}\eta^5\cdots$$

$$f''(\eta)=A_2+A_3\eta+\frac{A_4}{2}\eta^2+\frac{A_5}{6}\eta^3+\frac{A_6}{24}\eta^4+\cdots$$

$$f'''(\eta)=A_3+A_4\eta+\frac{A_5}{2}\eta^2+\frac{A_6}{6}\eta^3+\cdots$$

Therefore,

$$2\frac{d^3f}{d\eta^3}+f\frac{d^2f}{d\eta^2}=0$$

$$\Rightarrow 2\left(A_3+A_4\eta+\frac{A_5}{2}\eta^2+\frac{A_6}{6}\eta^3+\cdots\right)+\left(\frac{A_2}{2!}\eta^2+\frac{A_3}{3!}\eta^3+\cdots\right)$$

$$\times\left(A_2+A_3\eta+\frac{A_4}{2}\eta^2+\frac{A_5}{6}\eta^3+\frac{A_6}{24}\eta^4+\cdots\right)=0$$

$$\Rightarrow\left(2A_3+2A_4\eta+A_5\eta^2+\frac{A_6}{3}\eta^3+\cdots\right)+\left(\frac{A_2^2}{2}\right)\eta^2+\left(\frac{A_2A_3}{4}+\frac{A_2A_3}{2}\right)\eta^3+\cdots=0$$

$$\Rightarrow 2A_3+2A_4\eta+\left(A_2^2+2A_5\right)\frac{\eta^2}{2}+\left(\frac{A_6}{3}+\frac{3A_2A_3}{4}\right)\eta^3+\cdots=0.$$

Since, coefficients of various powers of η vanish separately,

$$A_3=0,\ A_4=0,\ A_5=-\frac{1}{2}A_2^2,\ A_6=0,\cdots$$

$$f(\eta)=\frac{A_2}{2!}\eta^2-\frac{1}{2}\frac{A_2^2}{5!}\eta^5+\frac{1}{4}.\frac{11}{81}.A_2^3\eta^8+\cdots$$

$$=A_2^{1/3}\left\{\frac{1}{2!}\left(A_2^{1/3}\eta\right)^2-\frac{1}{2}\frac{1}{5!}\left(A_2^{1/3}\eta\right)^5+\frac{1}{4}.\frac{11}{81}.\left(A_2^{1/3}\eta\right)^8+\cdots\right\}=A_2^{1/3}F\left(A_2^{1/3}\eta\right)$$

where,

$$F\left(A_2^{1/3}\eta\right)=\frac{1}{2!}\left(A_2^{1/3}\eta\right)^2-\frac{1}{2}.\frac{1}{5!}\left(A_2^{1/3}\eta\right)^5+\cdots$$

Now by the boundary condition that $f'\to 1$ at $\eta\to\infty$, we get,

$$1=\lim_{\eta\to\infty}\left[A_2^{2/3}F'\left(A_2^{1/3}\eta\right)\right]$$

$$\text{i.e., } A_2=\left[\frac{1}{\lim\limits_{\eta\to\infty}F'(\eta)}\right]^{3/2}$$

The value of A_2 can be obtained numerically. Howarth found that $A_2 = .332$. This completes the solution which is also known as Blasius solution.

5.2.4 Shearing stress on the plate

The shearing stress on the surface of the plate can be calculated with the help of the above solution. The shearing stress is given by

$$\tau_{\omega} = \mu\left(\frac{\partial u}{\partial y}\right)_{y=0} = \mu U\left(\frac{U}{\nu x}\right)^{1/2} f''(0) = 0.332\,\mu U\left(\frac{U}{\nu x}\right)^{1/2}$$

5.2.5 Boundary layer thickness

Although the velocity u reaches the potential value U asymptotically, a value which is very near to U is attained within a small distance δ. A measure of this boundary layer thickness is introduced by the following relation

$$U\delta = \int_0^{\infty}(U-u)\,dy\,.$$

The right hand side signifies the decrease in the flow rate due to friction within the boundary layer and the L.H.S. represents the total potential flow that has been displaced from the wall. So δ represents the distance to which the free stream has been displaced due to boundary layer. This δ is called displaced thickness. From a flat plate this given by

$$\delta = \int_0^{\infty}\left(1-\frac{u}{U}\right)dy\,.$$

The upper limit of integration is taken as $y = \infty$, because the integrand becomes zero outside the boundary layer.

5.3 Model Questions

Short Questions :

1. What is the difference between an ideal (non-viscous) and a real (viscous) fluid?
2. What is meant by dissipation?
3. Discuss the concept of boundary layer.
4. Define boundary layer thickness. What is its significance?

Broad Questions :

1. Deduce the vorticity equation for an incompressible viscous fluid.
2. Show that a viscous liquid cannot move without dissipation of energy by viscosity unless it moves as if rigid.
3. Discuss the motion of an incompressible viscous fluid through (i) a tube of circular, annular, elliptic and rectangular cross-section.
4. Deduce the equations of motion for two-dimensional boundary layer over a plane wall.
5. Find the Blasius solution for the two-dimensional boundary layer flow over a flat plate.

5.4 Summary

In this chapter, some properties of an incompressible viscous fluid are introduced and the motion of this fluid through tubes of different cross-section has been discussed. The concept of boundary layer and its property are also outlined.

NOTES

NOTES

NOTES