PREFACE

In the auricular structure introduced by this University for students of Post- Graduate degree programme, the opportunity to pursue Post-Graduate course in Subject introduced by this University is equally available to all learners. Instead of being guided by any presumption about ability level, it would perhaps stand to reason if receptivity of a learner is judged in the course of the learning process. That would be entirely in keeping with the objectives of open education which does not believe in artificial differentiation.

Keeping this in view, study materials of the Post-Graduate level in different subjects are being prepared on the basis of a well laid-out syllabus. The course structure combines the best elements in the approved syllabi of Central and State Universities in respective subjects. It has been so designed as to be upgradable with the addition of new information as well as results of fresh thinking and analysis.

The accepted methodology of distance education has been followed in the preparation of these study materials. Co-operation in every form of experienced scholars is indispensable for a work of this kind. We, therefore, owe an enormous debt of gratitude to everyone whose tireless efforts went into the writing, editing and devising of a proper lay-out of the materials. Practically speaking, their role amounts to an involvement in invisible teaching. For, whoever makes use of these study materials would virtually derive the benefit of learning under their collective care without each being seen by the other.

The more a learner would seriously pursue these study materials the easier it will be for him or her to reach out to larger horizons of a subject. Care has also been taken to make the language lucid and presentation attractive so mat they may be rated as quality self-learning materials. If anything remains still obscure or difficult to follow, arrangements are there to come to terms with them through the counselling sessions regularly available at the network of study centres set up by the University.

Needless to add, a great deal of these efforts is still experimental-in fact, pioneering in certain areas. Naturally, there is every possibility of some lapse or deficiency here and there. However, these do admit of rectification and further improvement in due course. On the whole, therefore, these study materials are expected to evoke wider appreciation the more they receive serious attention of all concerned.

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PG (MT)–XA (I)
Advanced Differential Geometry

UNIT - 1

§ 1.1 Calculus on R^n :

Let R denote the set of real numbers. For an integer n > 0, let R^n be the cartesian product

$$\underbrace{R\times R\times R\times\cdots\cdots\times R}_{\text{n times}}$$

of the set of all ordered n-tuples $(x^1,\cdots,x^n)$ of real numbers. Individual n-tuple will be denoted at times by a single letter, e.g. $x=(x^1,\cdots,x^n), y=(y^1,\cdots,y^n)$ and so on.

Co-ordinate functions : Let $x=(x^1,x^2,_i x^n)\in R^n$. Then, the functions $u_i:\ R^n\to R$ defined by $u_i(x^1,x^2,\cdots x^i\cdots x^n)=x^i$

We are now going to define a function to be differentiable of class C^∞.

A real-valued function $f:U\ C\ R^n\to R$,

U being an open set of R^n, is said to be of class c^k if

i) all its partial derivatives of order less than or equal to k exist and

ii) are continuous functions at every point of U.

By class C^0, we mean that f is merely continuous from U to R. By class C^∞, we mean that that partial derivatives of all orders of f exist and are continuous at every point of U. In this case, f is said to be a smooth function.

Note : By class C^ω on U, we mean that f is real analytic on U i.e. expandable in a power series about each point on U. A C^ω function is a C^∞ function but the converse is not true.

Exercise : 1. Let $f:R\to R$ be defined by

$$f(x)=e^{-\frac{1}{x^2}},\qquad x\neq 0$$
$$=0,\qquad x=0$$

Show that f is a differentiable function of class C^∞.

Solution : Note that

$$f'(o)=\lim_{h\to 0}\frac{f(o+h)-f(o)}{h}=\lim_{h\to 0}\frac{e^{-\frac{1}{h^2}}}{h}$$

Apply L'Hospital's Rule, on taking, $h=\frac{1}{u}$ we see that $h\to o$ gives $u\to\infty$

$$\therefore \; f'(o) = \lim_{u \to \infty} u.e^{-u^2}$$

$$= \lim_{u \to \infty} \frac{u}{e^{u^2}} \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{u \to \infty} \frac{1}{2ue^{u^2}}$$

$$= \lim_{u \to \infty} \frac{e^{-u^2}}{2u}$$

$$= 0$$

Again, $f'(x) = 2x^{-3}e^{-\frac{1}{x^2}}, \quad x \neq 0$

$\therefore \; f''(o) = \lim_{h \to 0} \dfrac{f'(0+h) - f'(0)}{h}$ and on putting $\dfrac{1}{h} = u,$ we get

$$f''(o) = \lim_{u \to \infty} \frac{2u^4}{e^{u^2}} \left(\frac{\infty}{\infty}\right)$$

Applying L' Hospital rule successively, we find

$$f''(0) = \lim_{u \to \infty} \frac{8u^3}{2ue^{u^2}}$$

$$= \lim_{u \to \infty} \frac{4u^2}{e^{u^2}}$$

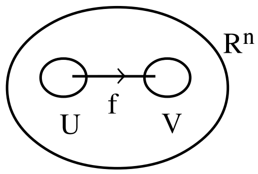
$$= \lim_{u \to \infty} \frac{8u}{2ue^{u^2}}$$

$$= \lim_{u \to \infty} \frac{4}{e^{u^2}}$$

$$= 0$$

Proceding in this manner, we can show that.

$f^n(0)=0$, for $n=1, 2, \cdots$



Hence f is a function of class C^∞.

A mapping $f: U \to V$

of an open set $U \subset R^n$ to an open set $V \subset R^n$ is called a <u>homeomorphism</u> if

i) f is bijective i.e. one to one and onto, as well as

ii) f, f^{-1} are continuous.

Exercise : 2. Let $f: R \to R$ be such that

$$f(x) = 5x+3$$

Show that f is a homoeomorphism on R.

3. Let $f: R \to R$ be defined by

$$f(x) = x^3$$

Test i) whether f is a differentiable function of class C^∞ or not

ii) whether f is a homeomorphism or not. [**Ans.** : i) f is of class C^∞.

ii) f is homeomorphism]

Solution : 2. Note that

$f(x)-f(y)=5(x-y)$

$\therefore$ $f(x)=f(y)$ if and only if $x=y$

Hence f is one one.

Let $y=5x+3$

$$\therefore \quad x=\frac{y-3}{5}$$

and hence $f^{-1}: R \to R$ is defined as

$$f^{-1}(y)=\frac{y-3}{5}$$

Again, $f\left(f^{-1}(y)\right)=y$ and $f^{-1}\left(f(x)\right)=x$, Thus f is onto.

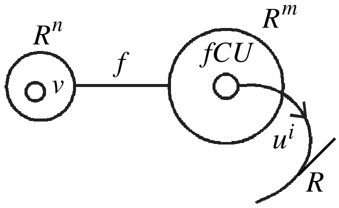
Consequently f is bijective.

Both f, f^{-1} are continuous functions, (being polynomial functions) f is a homeomorphism on R.

Note : (i) If $f : U \subset R^n \to R^m$ is a mapping, such that

$$f(x^1, \cdots x^n) = \left(f^1(x^1, \cdots, x^n), \cdots, f^m(x^1, \cdots, x^n)\right)$$

where $f^j(x) = u^j \circ f,\ 1 \le j \le m,\ u^j$ being co-ordinate functions on R^m



we define the <u>Jacobian matrix</u> of f at $(x^1, \cdots, x^n)$, denoted by J, as

$$J = \begin{pmatrix} \dfrac{\partial f^1}{\partial x^1} & \dfrac{\partial f^1}{\partial x^2} & \cdots & \dfrac{\partial f^1}{\partial x^n} \\ \dfrac{\partial f^2}{\partial x^1} & \dfrac{\partial f^2}{\partial x^2} & \cdots & \dfrac{\partial f^2}{\partial x^n} \\ \cdots & \cdots & \cdots & \cdots \\ \dfrac{\partial f^m}{\partial x^1} & \dfrac{\partial f^m}{\partial x^2} & \cdots & \dfrac{\partial f^m}{\partial x^n} \end{pmatrix}$$

(ii) In particular, when $m = n$ i.e., if $f : U \subset R^n \to R^n$ is a mapping such that,

if $f = (f^1, f^2, \cdots, f^n)$ has continuous partial derivatives i.e. if each f^i $i = 1, 2, \cdots, n.$ has continuous partial derivatives on U, we say that f is continuously differentiable on $U \subset R^n$.

(iii) If $f = (f^1, \cdots, f^n)$ is continuously differentiable on $U \subset R^n$ and the Jacobian is non-zero, then f is one-one on U.

Exercise : 4. Consider the mapping

$$\phi :\ R^2 \to R^2$$

given by

$$\phi : \quad y^1 = x^1 \cos x^2$$
$$y^2 = x^1 \sin x^2$$

Show that ϕ is one-to-one on a sufficiently small neighbourhood of each point (x^1, x^2) of R^2 with $x^1 \neq 0$.

Solution : The given mapping

$$\phi = (\phi^1, \phi^2) : R^2 \to R^2 \text{ is given by } \phi^1 = x^1 \cos x^2,\ \phi^2 = x^1 \sin x^2$$

Then, we have

$$\frac{\partial \phi^1}{\partial x^1} = \cos x^2,\ \frac{\partial \phi^1}{\partial x^2} = -x^1 \sin x^2,\ \frac{\partial \phi^2}{\partial x^1} = \sin x^2,\ \frac{\partial \phi^2}{\partial x^2} = x^1 \cos x^2$$

Hence each $\frac{\partial \phi^i}{\partial x^j}$, $i, j = 1, 2$ is continuous for all values of x^1 and x^2 in R^2. Thus ϕ is continuously differentiable on R^2.

Again the Jacobian is given by

$$J = \begin{vmatrix} \frac{\partial \phi^1}{\partial x^1} & \frac{\partial \phi^1}{\partial x^2} \\ \frac{\partial \phi^2}{\partial x^1} & \frac{\partial \phi^2}{\partial x^2} \end{vmatrix} = x^1 \neq 0 \text{ if and only if } x^1 \neq 0 \text{ in } R^2.$$

Consequently, ϕ is one-to-one on a sufficiently small neighbourhood of each point (x^1, x^2) of R^2 with $x^1 \neq 0$.

A mapping

$$f : U \to V$$

of an open set $U \subset R^n$ onto an open set $V \subset R^n$ is called a <u>C^k – diffeomorphism</u>, $k \geq 1$ if

i) f is a homeomorphism of U onto V and

ii) f, f^{-1} are of class C^k.

when f is a C^∞ – diffeomorphism, we simply say <u>diffeomorphism</u>.

Exercise : 5. Let $\phi : R^2 \to R^2$ be defined by

$$\phi(u, v) = (ve^u, u)$$

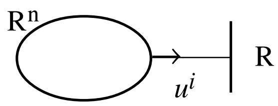
Determine whether ϕ is a diffeomorphism or not.

6. Let $\phi : R^2 \to R^2$ be defined by

$$\phi(x^1, x^2) = (x^1 e^{x^2} + x^2,\ x^1 e^{x^2} - x^2)$$

Show that ϕ is a diffeomorphism. [**Ans.** : 5. ϕ is a diffeomorphism]

For $i = 1, \cdots, n$; let $u^i : R^n \to R$



be the coordinate functions on R^n i.e. for every $p \in R^n$

1. 1) $u^i(p) = p^i$ where $p = (p^1, \cdots, p^n)$

Such $u^i s$ are continuous functions from $R^n \to R.$. We call this n-tuple of functions $(u^1, u^2, \cdots, u^n)$ the <u>standard co-ordinate system</u> of R^n.

If $f : U \subset R^n \to R^n$

is a mapping defined on U $\subset$ R^n, then, f is determined by its co-ordinate functions $(f^1, \cdots, f^n)$ where

1.2) $f^i = u^i \circ f \qquad , \ i = 1, \cdots, n$

and each $f^i : U \subset R^n \to R$ are real valued functions, defined on an open subset U of R^n.

Thus for every p $\in$ U $\subset$ R^n

$f^i(p) = (u^i \circ f)(p) = u^i\left(f(p)\right)$ where $f(p) = q = (q^1, \cdots, q^n)$

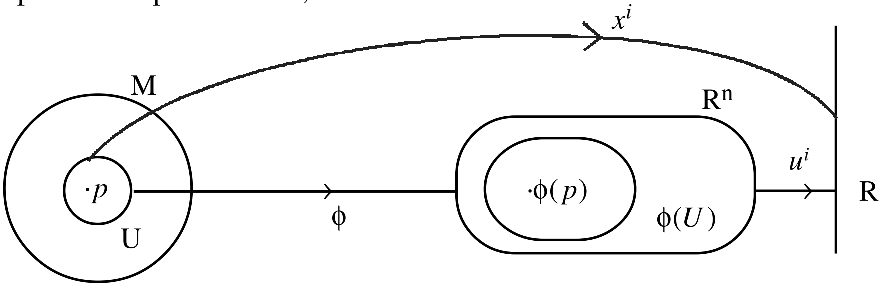
$= u^i(q^i, \cdots, q^n) = q^i$ by 1.1)

1.3) consequently $f(p) = \left(f^1(p), f^2(p), \cdots, f^n(p)\right), \ \forall\ p \in U \subset R^n$

The map f is of class c^k if each of its co-ordinate functions $f^i : i = 1, \cdots, n$ is of class c^k.

§ 1.2 Differentiable Mainfold :

Let M be a Hausdorff, second countable space. If every point of M has a neighbourhood homeomorphic to an open set in R^n, then



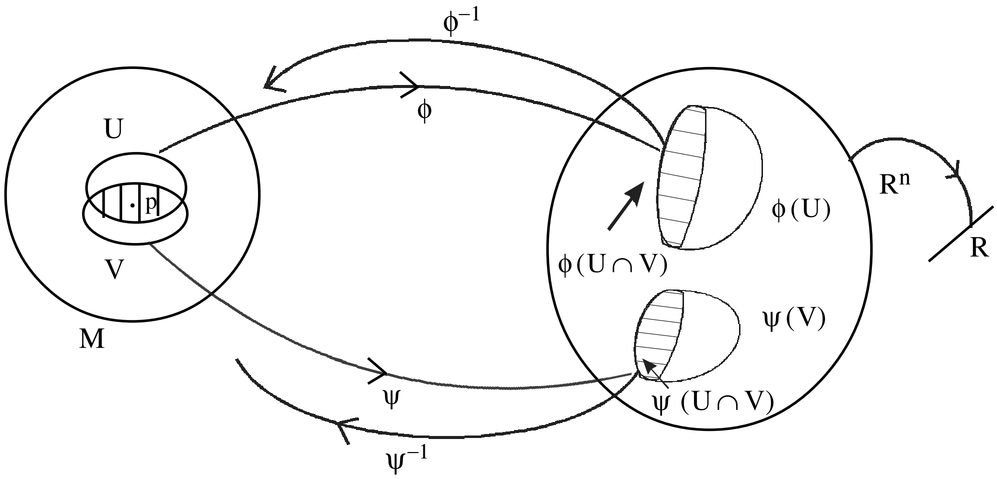
M is said to be a manifold. Thus in a manifold for each $p \in M$, there exists a neighbourhood U of $p \in M$ and a homeomorphism ϕ of U onto an open subset of R^n. The pair (U, ϕ) is called a <u>chart</u>.

Each such chart (U, ϕ) on M induces a set of n real valued functions on U defined by

2.1) $x^i = u^i \circ \phi$, $i = 1, 2, \cdots n$

where u^i, s are defined by (1.1) and it is to be noted that whatever be the point p and the neighbourhood $U, u^i, i = 1, 2, \cdots n$ always represent co-ordinate functions. The functions $(x^1, x^2 \cdots, x^n)$ are called <u>coordinate functions</u> or a coordinate system on U and U is called the domain of the coordinate system. The chart (U, ϕ) is sometimes called an <u>n-coordinate chart</u>.

Let (V, ψ) be another chart of p, which overlaps the previous chart (U, ϕ). Let $(y^1, \cdots, y^n)$ be local coordinate system on V of p, so that



2.2) $y^i = u^i \circ \psi, \quad , i = 1, 2, \cdots, n$

We can construct two composite maps

2.3) $$\phi \cdot \psi^{-1} : \psi\,(U \cap V) \subset R^n \to \phi\,(U \cap V) \subset R^n$$

$$\phi \cdot \psi^{-1} : \phi\,(U \cap V) \subset R^n \to \psi\,(U \cap V) \subset R^n$$

If these maps are of class c^k, we say that the two charts (U, ϕ) and (V, ψ) are c^k-related. If $q \in \phi\,(U \cap V)$ and

$$g : \phi\,(U \cap V) \subset R^n \to \psi\,(U \cap V) \subset R^n$$

is a mapping defined on an open set in R^n, we write

2.4) $g(q) = \psi(\phi^{-1}(q))$.

Exercise : 1 Find a functional relation between the two local coordinate systems defined in the overlap region of any point of a manifold M.

Solution : given that

$$q \in \phi\,(U \cap V),$$

$$g(q) = (\psi \circ \phi^{-1})\,(q) \text{ by 2.4)}$$

Let $\phi(p) = q$, where $p \in U \cap V$. Then

$$g(\phi(p)) = (\psi \circ \phi^{-1})(\phi(p)) = \psi(p)$$

or $\quad u^i\,(g(\phi(p))) = u^i\,(\psi(p)),\ i = 1, 2, \cdots, n$

or $\quad g^i(\phi(p)) = \psi^i(p)$ by 1.1)

or $\quad g^i(x^1(p), \cdots, x^n(p)) = y^i(p)$, as

$$x^i(p) = u^i(\phi(p)) = \phi^i(p)$$

$\therefore\ \phi(p) = (\phi^1(p)), \cdots, \phi^n(p) = (x^1(p), \cdots, x^n(p))$ and

$$y^i(p) = u^i\,(\psi(p)) = \psi^i(p)\ \ i = 1, 2, \cdots n.$$

consequently,

$$y^i = q^i(x^1, x^2, \cdots, x^n)$$

Note : If we consider

$$g(q) = \phi\,(\psi^{-1}\,(q)),$$

then one finds $x^i = g^i(y^1, y^2, \cdots, y^n), \quad i = 1, \cdots, n$

A collection $\Omega = \{(U_i, \phi_i)\}, i \in A,$ (an index set) of c^k related charts are said to be maximal collection if a co-ordinate pair (V, ψ), c^k related with every chart is also a member of Ω.

A maximal collection of c^k-related charts is called a <u>c^k-atlas</u>. A c^k n-dimensional differentiable manifold M is an n-dimensional manifold M together with a c^k-atlas.

Unless otherwise stated, we shall consider a <u>differentiable manifold of class</u> C^∞.

Examples : 1. R^n with the usual topology is an example of a differentiable manifold with respect to the atlas (U, ϕ) where U = R^n and ϕ = the identity transformation.

2. Let S^1 be the circle in the *xy* plane R^2, centered at the origin and of radius 1. We give S^1, the topology of a subspace of R^2. Let

$$U_1 = \{p = (x, y) \in s^1 | y > 0\}$$

$$U_2 = \{p = (x, y) \in s^1 | y < 0\}$$

$$U_3 = \{p = (x, y) \in s^1 | x > 0\}$$

$$U_4 = \{p = (x, y) \in s^1 | x < 0\}$$

Then each U_i is an open subset of S^1 and $S' = \underset{i}{U} U_i, i = 1, 2, 3, 4$

Let I = (–1, 1) be an open interval of R and we define

$\phi_1 : U_1 \to$ I $\subset$ R be such that

$\phi_1(x, y) = x$ i.e. $\phi_1^{-1}(x) = (x, y), y > 0$

$\phi_2 : U_2 \to$ I $\subset$ R be such that

$\phi_2(x, y) = x$ i.e. $\phi_2^{-1}(x) = (x, y), y < 0$

$\phi_3 : U_3 \to$ I $\subset$ R be such that

$\phi_3(x, y) = y$ i.e. $\phi_3^{-1}(y) = (x, y), x > 0$

$\phi_4 : U_4 \to$ I $\subset$ R be such that

$\phi_4(x, y) = y$ i.e. $\phi_4^{-1}(y) = (x, y), x < 0$

Note that each ϕ_i is a homeomorphism on R and thus each (u_i, ϕ_i) is a chart of S'. Now

$U_1 \cap U_2 = \phi$, $U_1 \cap U_3 = 1^{st}$ quadrant, $U_1 \cap U_4 = 2^{nd}$ quadrant, $U_2 \cap U_3 = 4^{th}$ quadrant, $U_2 \cap U_4 = 3^{rd}$ quadrant.

Then

$A = \{(U_i, \phi_i) : i = 1, 2, 3, 4)$ is an atlas of s^1

As $U \cap U_3 \neq \phi$, let $p \in U_1 \cap U_3$, then

$(\phi_1 \circ \phi_3^{-1})(y) = \phi_1(x, y) = x$ and

$(\phi_3 \circ \phi_1^{-1})(x) = \phi_3(x, y) = y$

Thus each $\phi_1 \circ \phi_3^{-1}$ and $\phi_3 \circ \phi_1^{-1}$ is of class C^∞. Similarly, it can be shown that each $\phi_1 \circ \phi_4^{-1}$, $\phi_4 \circ \phi_1^{-1}$, $\phi_2 \circ \phi_3^{-1}$, $\phi_3 \circ \phi_2^{-1}$, $\phi_2 \circ \phi_4^{-1}$, $\phi_4 \circ \phi_2^{-1}$, is of class C^∞ and hence s^1 is an one dimensional differentiable manifold with an atlas $\{(U_i, \phi_i)\}_{i=1,2,3,4}$

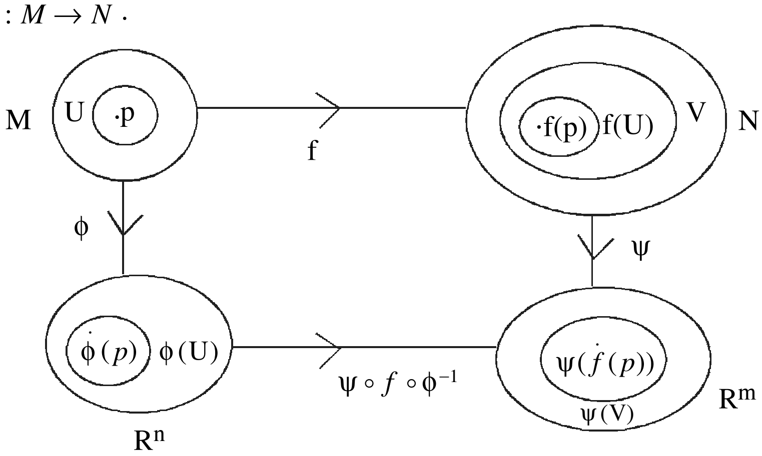
Exercise : 2. Let (M^n, A) be a differentiable manifold with a C^∞ atlas A. Let $p \in M$. Then there exists $(U, \phi) \in A$ such that $p \in U$ and $\phi(p) = 0$.

Note : 1. It is to be noted that every second countable, Hausdorff Space M admits partitions of unity. Partitions of unity admits Riemannian metric. Our aim is to study a Riemannian Manifold and for this reason we consider such topological spaces for a manifold.

2. It is enough to consider only a topological space for studying mainfold.

§ 1.3. Differentiable Mapping :

Let M be an n-dimensional and M be an m-dimensional differentiable manifold. A mapping $f : M \to N$.



is said to be a differentiable mapping of class c^k, if for every chart (U, ϕ) containing p of M and every chart (V, ψ) containing f(p) of N

3.1) i) $f(U) \subset V$ and

ii) the mapping $\psi \circ f \circ \phi^{-1}: \phi(U) \subset R^n \to \psi(V) \subset R^m$ is of class c^k.

By a <u>differentiable mapping</u>, we shall mean, unless otherwise stated, <u>a mapping of class</u> C^∞.

If $(x^1, \cdots, x^n)$ and $(y^1, \cdots, y^m)$ are respectively the local coordinate systems defined in a neighbourhood U of p of M and V of f(p) of N, then it can be shown, as done earlier

3.2) $y^j \circ f = g^j(x^1, \cdots, x^n), \quad j = 1, \cdots, m$

where g is a differentiable function defined on $V \subset N$ and

3.3) $g(q) = (\psi \circ f \circ \phi^{-1})(q), \quad q \in \phi(U).$

Let M and N be two n-dimensional differentiable manifolds. A mapping

$$f : M \to N$$

is called a <u>diffeomorphism</u> if

i) f and f^{-1} are differentiable mappings of class C^∞

ii) f is a bijection

In such cases, <u>M and N are said to be diffeomorphic to each other</u>.

Exercise : 1. Let M and N be two differentiable manifolds with M=N=R. Let (U, ϕ) and (V, ψ) be two charts on M and N respectively, where

U = R

$\phi : U \to R$ be the identity mapping and

V = R

$\psi : V \to R$ be the mapping defined by

$\psi(x) = x^3.$

Show that the two structures defined on R are not C^∞-related even though M and N are diffeomorphic where

$$f : M \to N$$

is defined by

$$f(t) = t^{1/3}$$

Hint : Note that, $(\psi \circ f \circ \phi^{-1})(x) = x$ and $(\phi \circ \psi^{-1})(x) = x^{1/3}$. Thus $\phi \circ \psi^{-1}$ is of class C^{∞} but $\phi \circ \psi^{-1}$ is not of class C^{∞}. Again

$$(\psi \circ f \circ \phi^{-1})(x) = x$$

Also $f(y) = f(x)$ if and only if $y = x$. Thus f is one-one. Finally

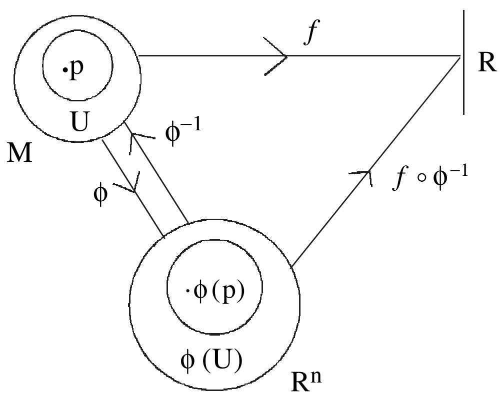
$$f^{-1}(y) = y^3, \text{ so that}$$

$f\left(f^{-1}(y)\right) = y$ and $f^{1}\left(f(x)\right) = x$. Thus f is a bijection.

Note : A diffeomorphism f of M onto itself is called a <u>transformation</u> of M.

A real-valued function on M ; i.e.

$f : \mathrm{M} \to \mathrm{R}$



is said to be a <u>differentiable function of class</u> C^{∞}, if for every chart (U, ϕ) containing p of M, the function

3.4) $f \circ \phi^{-1} : \phi(U) \subset R^n \to R$

is of class C^{∞}.

We shall often denote by F(M), the set of all differentiable functions on M and will sometimes denote by F(p), the set of functions on M which are differentiable at p of M.

It is to be noted that such F(M) is

i) an algebra over R

ii) a ring over R

iii) an associative algebra over R and

iv) a module over R

Where the defining relations are

a) $(f+g)(p) = f(p) + g(p)$

b) $(fg)(p) = f(p)g(p)$

c) $(\lambda f)(p) = \lambda f(p), \quad \forall\, f, g \in F(M),\ \lambda \in R,\ \ p \in M.$

§ 1.4. Differentiable Curve :

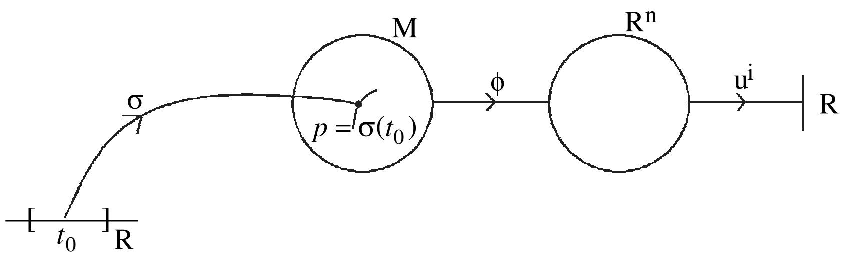
We are now in a position to define a curve on a manifold.

A differentiable curve through p in M of class C^r is a differentiable mapping

$\sigma : [a, b] \subset R \to M$, namely the restriction of a differentiable mapping of class C^r of an open interval] c, d [containing [a, b].

such that

4.1) $\sigma(t_0) = p \quad , \ a \le t_0 \le b$



Also

4.2) $(x^i \circ \sigma)\,(t) = (u^i \circ \phi)\,(\sigma(t)) = u^i(\phi\,(\sigma\,(t))) = u^i(\sigma^1(t), \cdots, \sigma^n(t)) = \sigma^i(t)$

We write it as

4.3) $x^i(t) = \sigma^i(t)$

The tangent vector to the curve $\sigma(t)$ at p is a function

$$X_p : F(p) \to R$$

defined by

4.4) $X_p f = \left[\frac{d}{dt} f(\sigma(t))\right]_{t=t_0} = \left[\begin{matrix}\lim \\ h \to 0\end{matrix} \frac{f(\sigma(t+h) - f(\sigma(t)}{h}\right]_{t=t_0}$

where $\quad p = \sigma(t_0),\ f \in F(p)$

It can be shown that it satisfies

4.5) $X_p(af + bg) = a(X_p f) + b(X_p g)$: Linearity

4.6) $X_p(fg) = g(p) X_p f + f(p) X_p g,\ \ f, g \in F(p)$: Leibnitz Product Rule.

Note : Each function X_p : F (p) $\to$ R, cannot be a tangent vector to some curve at $p \in M$, unless it is a linear function and satisfies Leibnitz Product Rule.

Exercises : 1. Let a curve σ on R^n be given by

$$\sigma^i = a^i + b^i t,\ \ i = 1, 2, \cdots, n$$

Find the tangent vector to the curve σ at the point (a^i).

2. If C is a constant function on M and X is a tangent vector to some curve σ at $p \in M$, then $X_p.C = 0$

[**Ans.** i) $(b^1, b^2, \cdots, b^n)$

ii) use 4.5), 4.6) and the definition of constant function.

Let us define

4.7) $(X_p + Y_p) f = X_p f + Y_p f$

4.8) $(bX_p) = bX_p f$, $b \in R$

If we denote the set of tangent vectors to M at p by $T_p(M)$, then from 4.7) and 4.8) it is easy to verify that $T_p(M)$ is a vector space over R. We are now going to determine the basis of such vector space.

For each i = 1, ... , n, we define a mapping

$$\frac{\partial}{\partial x^i} : F(p) \to R$$

by

4.9) $\left(\frac{\partial}{\partial x^i}\right)_p f = \left(\frac{\partial f}{\partial x^i(t)}\right)(p)$

Note that

$$\left(\frac{\partial}{\partial x^i}\right)_p (af+bg) = \left(\frac{\partial(af+bg)}{\partial x^i(t)}\right)(p) \qquad \text{by } 4.9) \ , \qquad a, b \text{ R}, f, g \in \text{F } (p)$$

$$= \left(a\frac{\partial f}{\partial x^i(t)}\right)(p) + \left(b\frac{\partial g}{\partial x^i(t)}\right)(p) \quad \text{by a) of 1.3}$$

$$= a\left(\frac{\partial f}{\partial x^i(t)}\right)(p) + b\left(\frac{\partial g}{\partial x^i(t)}\right) \qquad \text{by a) of 1.3}$$

$$= a\left(\frac{\partial}{\partial x^i}\right)_p f + b\left(\frac{\partial}{\partial x^i}\right) g$$

Thus such a mapping satisfies linearity property. It can be shown that

$$\left(\frac{\partial}{\partial x^i}\right)_p (fg) = g(p)\left(\frac{\partial}{\partial x^i}\right)_p f + f(p)\left(\frac{\partial}{\partial x^i}\right)_p g$$

Let us define a differentiable curve

$\sigma : [a, b] \subset \text{R} \rightarrow \text{M}$

by

4.10) $$\begin{cases} \sigma^i(t) = \sigma^i(t_0) + t & \text{, for fixed } i \\ \sigma^i(t) = 0, \quad j = 1, 2, \cdots, i-1, i+1, \cdots, n \end{cases}$$

then

$$\left[\frac{d}{dt} f(\sigma(t))\right]_{t=t_0} = \left\{\sum_{i=1}^{n} \frac{\partial f(\sigma(t))}{\partial \sigma^i(t)} \cdot \frac{d\sigma^i(t)}{dt}\right\}_{t=t_0} \quad \text{by chain rule}$$

$$= \left(\frac{\partial f}{\partial x^i(t)}\right)_{\sigma(t_0)} \quad \text{for fixed } i, \text{ by (4.3)}$$

$$= \frac{\partial f}{\partial x^i(t)}(p)$$

$$= \left(\frac{\partial}{\partial x^i}\right)_p f \quad \text{by} \quad (4.9)$$

Thus we can claim that each $\left(\frac{\partial}{\partial x^i}\right)$, $i = 1, 2, \cdots, n$ is a tangent vector to the curve σ defined above, at $p = \sigma(t_0)$.

Again from the definition of the tangent vector,

$$X_p f = \frac{d}{dt} f(\sigma(t))|_{t=t_0}$$

$$= \left\{\sum_{i=1}^{n} \frac{\partial f(\sigma(t))}{\partial \sigma^i(t)} \cdot \frac{d\sigma^i(t)}{dt}\right\}_{t=t_0} \qquad \text{by chain rule}$$

$$= \sum_{i=1}^{n} \left(\frac{dx^i(t)}{dt}\right)_{t=t_0} \frac{\partial f(\sigma(t_0))}{\partial x^i(t)} \qquad \text{by (4.3)}$$

$$= \sum_{i=1}^{n} \left(\frac{dx^i(t)}{dt}\right)_{t=t_0} \left(\frac{\partial}{\partial x^i(t)}\right)_p f$$

We write it as

4.11) $$X_p = \sum_{i=1}^{n} \xi^i(p) \left(\frac{\partial}{\partial x^i}\right)_p \quad \text{where}$$

4.12) $$\xi^i(p) = \left(\frac{dx^i(t)}{dt}\right)_{t=t_0}, \quad i = 1, \cdots, n$$

Thus each ξ^i : M $\to$ R, $i = 1, \cdots, n$ is a differentiable function and every tangent vector, say X_p, to some curve, say $\sigma(t)$ at $p = \sigma(t_0)$ can be expressed as a linear combination of the tangent vector $\frac{\partial}{\partial x^i(t)}$, $i = 1, \cdots, n$ to the curve σ defined in (4.10)

If possible, for a given linear combination of the form $\sum \xi^i(p)\left(\frac{\partial}{\partial x^i}\right)$, where ξ^i ,s are functions on M, let us define a curve σ by

$$\sigma : \sigma^i(t) = \sigma^i(t_0) + \xi^i(p)t\ ,\ a \le t_0 \le b$$

then it can be shown that the tangent vector to this curve is $\sum \xi^i(p)\left(\frac{\partial}{\partial x^i}\right)_p$

If we assume that

$$\sum \xi^i(p)\left(\frac{\partial}{\partial x^i}\right)_p = 0$$

then,

$$\sum_i \xi^i(p)\left(\frac{\partial}{\partial x^i}\right)_p x^k = 0 \text{ where } x^k : \mathrm{M} \to \mathrm{R},\ K = 1, 2, \cdots n.$$

or $\displaystyle\sum_i \xi^i(p)\left(\frac{\partial x^k}{\partial x^i}\right)_p = 0$

$\therefore\ \xi^k(p) = 0.$ for $k = 1, 2, \cdots n.$

Thus the set $\left\{\left(\frac{\partial}{\partial x^i}\right)_p : i = 1, \cdots, n\right\}$ is linearly independent. Hence we state

Theorem 1 : If $(x^1, \cdots, x^n)$ is a local coordinate system in a neighbourhood U of p $\in$ M, then, the basis of the tangent space $T_p(M)$ is given by

$$\left\{\left(\frac{\partial}{\partial x^1}\right)_p, \cdots, \left(\frac{\partial}{\partial x^n}\right)_p\right\}$$

Let us define $\mathrm{T(M)} = \bigcup_{p \in M} T_p(\mathrm{M})$. This T(M) is called the tangent space of M.

§ 1.5. Vector Field :

In classical notation, if to each point p of R^3 or in a domain U of R^3, a vector

$\alpha : p \to \alpha(p)$

is specified, then, we say that a vector field is given on R^3 or in a domain U of R^3.

A vector field X on M is a correspondance that associates to each point $p \in M$, a vector $X_p \in T_p(M)$. In fact, if $f \in F(M)$, then Xf is defined to be a real-valued function on M, defined as follows

5.1) $(Xf)(p) = X_p f$

A vector field X is called differentiable if Xf is so for every $f \in F(M)$. Using (4.11) of § 1.4, a vector field X may be expressed as

5.2) $$X = \sum \xi^i \frac{\partial}{\partial x^i}$$

where ξ^i's are differentiable functions on M.

Let $\chi(M)$ denote the set of all differentiable vector fields on M. We define

5.3) $$\begin{cases} (X+Y)f = Xf + Yf \\ (bX)f = b(Xf) \end{cases}$$

It is easy to verify that $\chi(M)$ is a vector space over R.

Also, for every $f \in F(M)$, fX is defined to be a vector field on M, defined as

5.4) $(fX)(p) = f(p)X_p$

Let us define a mapping as $[\ ,\ ] : F(M) \to F(M)$ as

5.5) $[X, Y]f = X(Yf) - Y(Xf), \quad \forall X, Y \in \chi(M)$

Such a bracket is known as **Lie bracket** of X, Y.

Exercises : 1. Show that for every X, Y, Z in $\chi(M)$, for every f, g in F(M),

i) $[X, Y] \in \chi(M)$

ii) $[bX, Y] = [X, bY] = b[X, Y], b \in R$

iii) $[X + Y, Z] = [X, Z] + [Y, Z]$

iv) $[X, Y + Z] = [X, Y] + [X, Z]$

v) $[X, X] = \theta$ vi) $[X, Y] = -[Y, X]$

vii) $\left[X,[Y,Z]\right] + \left[Y,[Z,X]\right] + \left[Z,[X,Y]\right] = \theta$: Jacobi Identity

viii) $[fX, gY] = (fg)[X, Y] + \{f(Xg)\}Y - \{g(Yf)X\}$

a) $[X, fY] = f[X, Y] + (Xf)Y$

b) $[fX, Y] = f[X, Y] - (Yf)X$

2. In terms of a local co-ordinate system

i) $\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^i}\right] = 0$

ii) $[X, Y] = \sum_{i,j}\left(\xi^i \frac{\partial \xi^j}{\partial x^i} - \zeta^i \frac{\partial \xi^j}{\partial x^i}\right) \frac{\partial}{\partial x^j}$, where $X = \xi^i \frac{\partial}{\partial x^i}$, $Y = \xi^j \frac{\partial}{\partial x^j}$

3. Complete [X, Y] where

i) $X = \frac{\partial}{\partial x^1}$, $Y = \frac{\partial}{\partial x^2} + e^{x1} \frac{\partial}{\partial x^3}$

ii) $X = x^1 x^2 \frac{\partial}{\partial x^1}$, $Y = x^2 \frac{\partial}{\partial x^2}$

4. Prove that

i) χ (M) is a F(M) module

Hints : 1. viii) Note that

$\{f(Yh)\}(p) = f(p)(Yh)_p$ by (5.4) of § 1.5)

$= f(p) Y_p{}^h$ by (5.1) of § 1.5)

Again, $\{(fY)\}(p) = (fY)(p)h$ by (5.1)

$= f(p) Y_p h$ by (5.4)

Thus $\{f(Yh)\}(p) = \{(fY)h\}(p), \forall p$

$f(Yh) = (fY)h$

Use the above result, 5.5) of § 1.5 & (4.6) of § 1.4, the result follows after a few steps.

§. 1.6. Integral Curve :

In this article, we are going to give the geometrical interpretation of a vector field.

Let Y be a vector field on M. The assignment of the vector Y_p

at each point $p \in U \subset M$, is given by

$Y : p \to Y_p \in T_p(M)$

A curve σ is an <u>integral curve</u> of Y if the range of σ is contained in U and for every $a \leq t_0 \leq b$ in the domain [a, b] of σ, the tangent vector to σ at $\sigma(t_0) = p$ coincides with Y_p i.e.

$$Y_p = Y_{\sigma(t_0)}$$

$$Y_p f = Y_{\sigma(t_0)} f \, , \qquad , \qquad \forall f \in F(M)$$

$$= \left[\frac{d}{dt}(f \circ \sigma)(t)\right]_{t=t_0} \qquad \text{by (4.4) of § 1.4}$$

Using 4.11) § 1.4 one can write

$$\sum_i \eta^i(p)\left(\frac{\partial}{\partial x^i}\right)_p f = \left[\frac{d}{dt}(f \circ \sigma)(t)\right]_{t=t_0} \quad \text{where } \eta^i\text{'s are functions on M.}$$

$$= \sum \left(\frac{dx^i(t)}{dt}\right)_{t=t_0} \left(\frac{\partial}{\partial x^i}\right)_p f$$

As $\left\{ \frac{\partial}{\partial x^i} : i = 1, \cdots, n \right\}$ are linearly independent, we must have

$$\eta^i(p) = \left(\frac{dx^i}{dt}\right)_{t=t_0}$$

or $\quad \eta^i(\sigma(t))_{t=t_0} = \left(\frac{dx^i}{dt}\right)_{t=t_0}$

or $\eta^i \, (\sigma^1(t), \sigma^2(t), \cdots, \sigma^n(t)_{t=t_0} = \left(\frac{dx^i}{dt}\right)_{t=t_0}$

Using (4.3) of § 1.4 we get

$$\eta^i \,(x^1(t), x^2(t), \cdots, x^n(t)_{t=t_0} = \left(\frac{dx^i}{dt}\right)_{t=t_0}$$

Hence they are related by

6.1) $\dfrac{dx^i}{dt} = \eta^i\big((x^1(t), \cdots, x^n(t)\big)$

Exercises : 1. Find the integral curve of a zero vector.

2. Find the integral curve of the following vector field

i) $X = x^1 \dfrac{\partial}{\partial x^1} + x^2 \dfrac{\partial}{\partial x^2}$ on R^2

ii) $X = e^{-x^1} \dfrac{\partial}{\partial x^1}$ on R

iii) $X = \dfrac{\partial}{\partial x^1} + (x^1)^2 \dfrac{\partial}{\partial x^2}$ on R^2

Solution : 2.i) From (6.1) of § 1.6, we see that

$$\frac{dx^1}{dt} = x^1, \quad \frac{dx^2}{dt} = x^2$$

or $\dfrac{dx^1}{x^1} = dt, \quad \dfrac{dx^2}{x^2} = dt$

Integrating

$\log x^1 = t + C$, $\log x^2 = t + D$ say, where C, D are integrating constant.

When t = 0, if $x^1 = p^1, \; x^2 = p^2,$ then from

$x^1 = Ce^t$ and $x^2 = De^t$

we find that

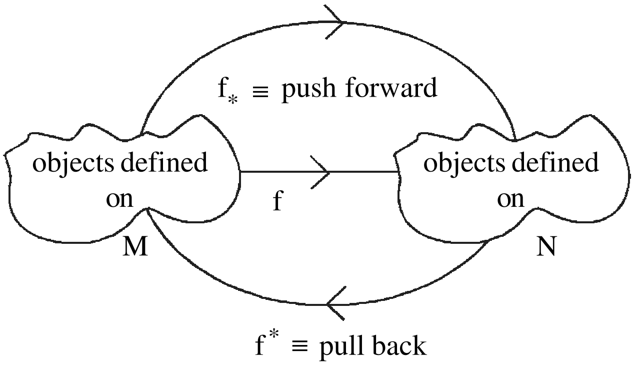
$p^1 = C, p^2 = D$

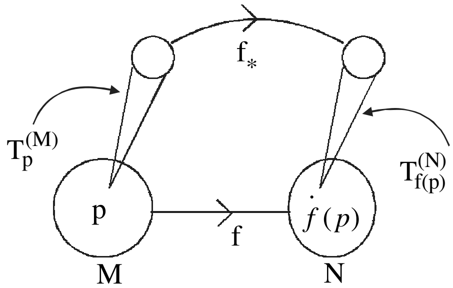
Thus $\gamma \; : \; \left(p^1 e^t, \; p^2 e^t\right)$ is the integral curve of X passing through the point (p^1, p^2)

§. 1.7 Differential of a mapping :

Let

f : M → N





be a a differentiable mapping of an n-dimensional manifold M to an m-dimensional manifold N. Let F(p) denote the set of all differentiable functions at p∈ M and $F(f(p))$ denote the set of all differentiable functions at $f(p) \in N$. Such a map f, induces a map

$f^*: F(f(p)) \to F(p)$, usually called pull back map.

and is defined by

7.1) $f^*(g) = g \circ f, \quad g \in F(f(p))$

called the <u>pull back</u> of g by f, which satisfies

7.2) $$\begin{cases} f^*(ag+bh) = a(f^*g) + b(f^*h) \\ f^*(gh) = f^*(g) f^*(h) \end{cases}$$ where $g, h \in F(f(p))$ and $a, b \in R$

The map f, also induces a linear mapping

$f_*: T_p(M) \to T_{f(p)}(N)$

such that

7.3) $$\left(f_*(X_p)\right) g = X_p(g \circ f) = X_p\left(f^*(g)\right)$$

called the push forward of X by f. Such f_* is also called <u>derived linear map</u> or <u>Jacobian map</u> or <u>differential map</u> of f <u>on $T_p(M)$</u>

Let us write

7.4) $f_*(X_p) = (f_* X)_{f(p)}$

We can also define push forward of X by f, geometrically, in the following manner :

Given a differential mapping

$f : M \to N,$

the differential of *f* at $p \in M$ is the linear mapping

$f_* : T_p(M) \to T_{f(p)}(N)$

defined as follows :

For each $X_p \in T_p$ (M), we choose a curve $\sigma(t)$ in M such that X_p is the tangent vector to the curve $\sigma(t)$ at $p = \sigma(t_0)$. Then $f_*(X_p)$ is defined to be the tangent vector to the curve $f(\sigma(t))$ at $f(p) = f(\sigma(t_0))$

Exercises :

1. If f is a differentiable map from a manifold M into another manifold N and *g* is a differentiable map from N into another manifold L, then, show that

i) $(g \circ f)_* = g_* \circ f_*$ ii) $(g \circ f)^* = f^* \circ g^*$

2. If f is a transformation of M and g is a differentiable function on M, prove that

i) $f_*[X, Y] = f_*[X, Y]$

ii) $f^*((f_* X)g) = X(f^* g)$

iii) $f_*(gX) = (g \circ f^{-1})(f_* X)$

for all vector fields X, Y on M.

Solution : 1. By definition, $f_*(X_p)$ is the tangent vector to the curve $f(\sigma(t))$ at $f(p) = f(\sigma(t_0))$ where X_p is the tangent vector to the curve $\sigma(t)$ at $p = \sigma(t_0)$. Hence by (4.4) of § 1.4

$$\left(f_*(X_p)\right)g = \left[\frac{d}{dt}g\left(f(\sigma(t))\right)\right]_{t=t_0} \quad g \in F\left(f(p)\right)$$

$$= \left[\frac{d}{dt}(g \circ f)\left(\sigma(t)\right)\right]_{t=t_0}$$

$$= X_p \ (g \circ f) \text{ by 4.4) of § 1.4}$$

Hints 3. Given that

f : M $\to$ M

is a transformation and hence for every p$\in$M, $f(p) = q$, say.

Thus, $p = f^{-1}(q)$

consequently, from 7.3) of § 1.7, we find that

$$\left\{\left(f_*(X_p)g\right\}f(p) = \left\{X_p(g \circ f)\right\}(p), \quad \forall p \in M$$

or $\quad \left\{\left(f_*(X_p)g\right)\right\}(q) = \left\{X_p(g \circ f)\right\}f^{-1}(q)$

or $\quad \left(f_*(X)\right)g = \left(X(g \circ f)\right)f^{-1}$

Using this relation, one can deduce the three results.

We are now going to give a matrix representation of the linear mapping f_*.

Theorem 1 : If f is a mapping from an n-dimensional manifold M to an m-dimensional manifold N, where $(x^1, \cdots, x^n)$ is the local co-ordinate system in a neighbourhood of a point p of M and $(y^1, \cdots y^m)$ is the local co-ordinate system in a neighbourhood of $f(p)$ of N, then

$$f_*\left(\frac{\partial}{\partial x^i}\right)_p = \sum_{j=1}^{m} \frac{\partial f^j}{\partial x^i}_p \left(\frac{\partial}{\partial y^j}\right)_{f(p)} \text{ where } f^j = y^j \circ f$$

Proof : We write

$$f_*\left(\frac{\partial}{\partial x^i}\right)_p = \sum_{j=1}^{m} a_i^j \left(\frac{\partial}{\partial y^j}\right)_{f(p)}, \ i =, \ldots, n$$

where a_i^j's are unknown to be determined

or $\left\{f_*\left(\frac{\partial}{\partial x^i}\right)\right\}y^k = \sum_{j=1}^{m} a_i^j \left(\frac{\partial}{\partial y^j}\right)_{f(p)} y^k$ where each $y^k = \in F(f(p))$ $k = 1,\ldots,m$

using 7.3) of § 1.7, we find

$$\left(\frac{\partial}{\partial x^i}\right)_p (y^k \circ f) = \sum_{j=1}^{m} a_i^j \delta_j^k$$

or $\left(\frac{\partial}{\partial x^i}\right)_p f^k = a_i^k$

or $\left(\frac{\partial f^k}{\partial x^i}\right)_p = a_i^k$ by (4.9) of § 1.4

Thus

$$f_*\left(\frac{\partial}{\partial x^i}\right)_p = \sum_{j=1}^{m}\left(\frac{\partial f^j}{\partial x^i}\right)_p \left(\frac{\partial}{\partial y^j}\right)_{f(p)}$$

Note : 1. The matrix of f_*, denoted by (f_*) is given by

$$(f_*) = \begin{pmatrix} \frac{\partial f^1}{\partial x^1} & \frac{\partial f^1}{\partial x^2}\cdots & \frac{\partial f^1}{\partial x^n} \\ \frac{\partial f^2}{\partial x^1} & \frac{\partial f^1}{\partial x^2}\cdots & \frac{\partial f^1}{\partial x^n} \\ \frac{\partial f^m}{\partial x^1} & \frac{\partial f^m}{\partial x^2} & \frac{\partial f^m}{\partial x^n} \end{pmatrix}$$

Note : 2. The kernel of f_* is the set of $X_p \in T_p(M)$ for which

$$f_*(X_p) = \theta$$

The image of f_* is the set of $Y_{f(p)} \in T_{f(p)}(N)$ for which, there exists $X_p \in T_p(M)$ such that

$$f_*(X_p) = Y_{f(p)}$$

Now from a known theorem

dim (kernel f_*) + dim (Range f_*) = dim $T_p(M)$.

We write it as

7.5) dim (kernel f_*) + dim (Range f_*) = dim $T_p(M)$ for each $p \in M$

The dim (Range f_*) is called the rank f_*

If rank f_* = dim $T_p(M)$ we say

i) f is an immersion if dim M $\leq$ dim N and f(M) is an immersed submanifold of N

ii) f is an imbedding if f is one to one and an immersion and then f(M) is an imbedded submanifold of N

iii) f is a submersion if dim M $\geq$ dim N.

Exercises : 1. Show that

$$f: R \to R^2$$

given by

f(t) = (a cost, sint)

is an immersion.

2. Find (f_*) in the following cases

i) f : $R^2 \to R^2$ given by f = $\left((x^1)^2 + 2(x^2)^2, 3x^1x^2\right)$

ii) f : $R^2 \to R^2$ given by f = $\left(x^1e^{x^2} + x^2, x^1e^{x^2} - x^2\right)$ at (0, 0)

where (x^1, x^2) are the local co-ordinates on R^2

§. 1.8 f-related vector Field :

Let X and Y be fields on M and N respectively.

Then, for $p \in M$, let $X_p \in T_p(M)$ and $Y_{f(p)} \in T_{f(p)}(N)$ and such that

8.1) $$f_*(X_p) = Y_{f(p)}$$

where $f: M \to N$ is a differentiable mapping and f_* is already defined in the previous article. In such a case, we say that the two vector fields X, Y are f-related.

For $g \in F(f(p))$ we see that

$$\{f_*(X_p)\}g = Y_{f(p)}^{g}$$

Using 7.3) of § 1.7 and (5.1) of § 1.5 we find that

$$X_p(g \circ f) = (Yg)f(p), \ \forall p$$

Then

8.2) $$X(g \circ f) = (Yg)f$$

If f is a transformation on M and

$$f_*(X_p) = X_{f(p)}$$

we say that, X is f-related to itself or X <u>is invariant under f</u>. We also write it as

8.3) $$f_* X = X$$

Exercises : 1. Let $X_i, Y_i (i = 1, \ 2)$ be two f-related vector fields on M and N respectively. Show that the vector fields $[X_1, X_2]$ and $[Y_1, Y_2]$ are also f-related.

2. Prove that two vector fields X, Y respectively on M and N are f-related if and only if

$$f^*((f_* X)g) = X(f^* g)$$

where f : M $\to$ N is a C^∞ map.

3. If f is a transformation on M, show that, for every $X \in \chi(M)$, there exists a unique f-related vector field to X.

Solution : 1. From the definition of the Lie bracket, we see that

$$\begin{aligned}
[X_1, X_2](g \cdot f) &= X_1(X_2(g \cdot f)) - X_2(X_1(g \cdot f)) \\
&= X_1((Y_2 g)f) - X_2((Y_1 g)f) \qquad \text{by (8.2) above} \\
&= \{Y_1(Y_2 g)\}f - \{Y_2(Y_1 g)\}f \qquad \text{by (8.2) above} \\
&= \{Y_1(Y_2 g) - Y_2(Y_1 g)\}f
\end{aligned}$$

$[X_1, X_2](g \circ f) = \{[Y_1, Y_2]g\}f$ from the definition of the Lie Bracket. Hence from 8.2), one claims that $[X_1, X_2]$ and $[Y_1, Y_2]$ are f-related.

§. 1.9 One parameter group of transformations on a manifold :

Definitioin

Let a mapping

$\phi : R \times M \to M$

is defined by

$\phi : (t, p) \to \phi_t(p)$

which satisfy

i) for each $t \in R$, $\phi(t, p) = \phi_t(p)$ is a transformation on M and $\phi_0(p) = p$

ii) for all t, s, t + s $\in$ R

$\phi_t(\phi_s(p)) = (\phi_t \circ \phi_s)(p) = \phi_{t+s}(p)$

Then the family $\{\phi_t | t \in R\}$ of mappings is called a one-parameter group of transformations on M.

Exercise : 1. Let $\{\phi_t | t \in R\}$ be a one-parameter group of mappings on M. Show that

i) $\phi_{-t} = (\phi_t)^{-1}$

ii) $\{\phi_t | t \in R\}$ form an abelion group.

Let us set

9.1) $\Psi(t) = \phi_t(p)$

Then $\Psi(t)$ is a differentiable curve on M such that

$\Psi(0) = \phi_0(p) = p$ by Def. (i) above

Such a curve is called the <u>orbit</u> through p of M. The tangent vector, say X_p to the curve $\psi(t)$ at p is therefore

9.2) $X_p f = \left[\dfrac{d}{dt} f(\Psi(t))\right]_{t=0} = \lim\limits_{t \to 0} \dfrac{f(\phi_t(p)) - f(p)}{t}$, $\forall f \in F(M)$

In this case, we say that $\{\phi_t | t \in R\}$ induces the vector field X and X is called the generator of $\{\phi_t\}$. The curve $\Psi(t)$ defined by 9.1) is called the integral curve of X.

Exercises : 2. Show that the mapping

$$\phi : R \times R^3 \to R^3$$

defined by

$$\phi(t, p) = (p^1 + t, p^2 + t, p^3 + t)$$

is a one-parameter group of transformations on M and the generator is given by $\frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2} + \frac{\partial}{\partial x^3}$

3. Let M = R^2 and let

$$\phi : R \times M \to M$$

be defined by

$$\phi(t, (x, y)) = (xe^{2t}, ye^{-3t})$$

Show that ϕ defines a one-parameter group of transformation on R^2 and find its generator.

Note : Since every 1-parameter group of transformations induces a vector field on M, the question now arises whether every vector field on M generates one parameter group of transformations. This question has been answered in the negative.

Example : Let

$$X = -e^{x^1} \frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2}$$

on M = R^2. Then,

$$\frac{dx^1}{dt} = -e^{x^1}, \quad \frac{dx^2}{dt} = 1$$

Thus $e^{-x^1} = t + A, \; x^2 = t + B,$ where A, B are integrating constant.

Let $x^1 = p^1, \; x^2 = p^2$ for $t = 0$ Then, $A = e^{-p^1}, \; B = p^2$.

Consequently the integral curve of X is $\psi(t) = \left(\log \frac{1}{t + e^{-p^1}}, t + p^2 \right)$

which is not defined for all values of t in R. Thus, if $\psi(t) = \phi_t(p)$, then, X does not generate one parameter group of transformations.

Problem 7 leads us to the following definition :

Let $I_\in$ be an open interval $(-\in, \in)$ and U be a nbd of p of M. Let a mapping

$\phi : I_\in \times U \to \phi_t(U) \subset M$

denoted by

$\phi(t, p) = \phi_t(p)$

be such that

i) U is an open set of M

ii) for each $t \in I_\in$, $\phi(t, p) \to \phi_t(p)$ is a transformation of U onto an open set $\phi_t(U)$ of M

and $\phi_0(p) = p$

iii) if t, s, t + s are in $I_\in$ and if $\phi_s(p) \subset$ U

$\phi_t(\phi_s(p)) = \phi_{t+s}(p)$

Such a family $\{\phi_t | t \in I_\in\}$ of mappings is called a local one parameter group of transformations, defined on $I_\in \times U$.

We are now going to establish the following theorem

Theorem 1 : Let X be a vector field on a manifold M. Then, X generates a local one-parameter group of transformations in a neighbourhood of a point of M.

Proof : Let $(x^1, x^2, ...x^n)$ be a local coordinate system in a neighbourhood U of p of M such that $\phi(p) = (0,...,0) \in R^n$, where (U, ϕ) is the chart at p of M. Then $x^i(p) = (u^i \circ \phi)(p) = 0$, $i = 1, ..., n$

Let

$$X = \sum_i \xi^i(x^1, ..., x^n) \frac{\partial}{\partial x^i}$$

be a given vector field on U, the neighbourhood of p $\in$ M, where each $\xi^{i,}s$ the components of X, are differentiable functioins on U of M. Then, for every X on M, we have a ϕ -related vector field on, $\phi(\mathrm{U}) = \mathrm{U}_1\mathrm{CR}^n$ with ϕ (p) = (0,...,0) $\in \mathrm{U}_1\mathrm{CR}^n$.

Let $\eta^{i,}$ s be the components of the ϕ -related vector field on U_1 of R^n. Then by the existence theorem of ordinary differential equations, for each ϕ (p)$\in$ $\mathrm{U}_1 \subset \mathrm{R}^n$, there exists a $\delta_1 > 0$ and a neighbourhood V_1 of ϕ (p), $\mathrm{V}_1 \subset \mathrm{U}_1$ such that, for each $q = (q^1,...q^n) \in \mathrm{V}_1$, $q = \phi(r)$, say, there exists n-tuple of $\overset{\infty}{C}$ functions $f^1(t,q),...f^n(t,q)$ defined on $\mathrm{I}_{\delta_1} \subset \mathrm{I}_{\in_1}$ and mapping $\mathrm{f}^i : I_{\delta_1} \to \mathrm{V}_1 \subset \mathrm{U}_1$, $i = 1,...,n$ which satisfies the system of first order differential equations

1) $\dfrac{df^i(t)}{dt} = \eta^i(t,\phi(p)), \quad i = 1,...,n$

with the initial condition

2) $f^i(0,q) = q^i$

Let us write

3) $\theta_t(q) = (f^1(t,q),...,f^n(t,q))$

We are going to show

$\theta_{t+s}(q) = \theta_t(\theta_s(q)).$

Note that if t, s, t + s are in I_{δ_1} and if $\theta_s(q) \in \mathrm{V}_1 \subset \mathrm{U}_1$ then each $f^i(t+s,q)$, $f^i(t,\theta_s(q))$ are defined on $\mathrm{I}_{\delta_1} \times \mathrm{U}_1$. Now let us set

$(g^1(t),...,g^n(t)) = (f^1(t+s,q),...f^n(t+s,q))$

For simplicity, we write

$(g^i(t)) = (f^i(t+s,q))$

Then each $g^i(t)$ is defined on $\mathrm{I}_{\delta_1} \times \mathrm{U}_1$ and hence satisfies 1) with the initial condition

4) $(g^i(o)) = (f^i(s,q))$

Also, let us set

$$\left(h^1(t),\ldots,h^n(t)\right)=\left(f^1(t,\phi_s(q),\ldots,f^n(t,\theta_s(q)\right)$$

For simplicity, we write

$$\left(h^i(t)\right)=\left(f^i(t,\theta_s(q)\right)$$

then each $h^i(t)$ is defined on $I_{\delta_1}\times U_1$ and hence satisfies 1) with the initial condition

$$\left(h^i(o)\right)=\left(f^i\left(o,\theta_s(q)\right)\right)$$

$$=\left(\theta_s(q)\right)^i \quad \text{by 2)}$$

$$=\left(f^i(s,q)\right) \quad \text{by 3)}$$

$$=\left(g^i(o)\right) \quad \text{by 4)}$$

Hence from the uniqueness we must have

$$\left(g^i(t)\right)=\left(h^i(t)\right)$$

Using 3) we must have

$$\theta_{t+s}(q)=\theta_t\left(\theta_s(q)\right).$$

Thus, we claim that, for every vector field defined in a neighbourhood U_1 of ϕ (p) of R^n, there exists $\left\{\phi_t \mid t\in I_{\delta_1}\right\}$ as its local 1-parameter group of transformations defined on $I_{\delta_1}\times U_1$.

Let us set

$V = \phi^{-1}(V_1) \subset U$

and define

$$\psi : I_\in \times V \to \psi_t(V)\subset M$$

as follows

$$\psi_t(r)=\phi^{-1}\left(\theta(t,q)\right)$$

Then

i) $\psi(t,r)\to\psi_t(r)$ is a transformation of V onto ψ_t(V) of M

ii) if t, s, t + s are in $I_\in$ and if $\psi_s(r) \subset$ V, then

$$\psi_t(\psi_s(r)) = \phi^{-1}(\theta(t, \phi(\psi_s(r)))$$

$$= \phi^{-1}(\theta(t+s, q)) \text{ , after a few steps}$$

$$= \psi_{t+s}(r)$$

Thus for the given vector field X, defined in a neighbourhood U of p of M, there exists $\{\psi_t \,|\, t \in I_\in\}$ as its local 1-parameter group of transformations, defined on $I_\in \times V \subset U$ of M. Note that if we define

$$\gamma(t) = \psi_t(r) = \phi^{-1}(\phi(t, q)), \quad q = \phi(r)$$

$$= \phi^{-1}(\sigma(t)), \text{ say,}$$

then $\phi^{-1}(\sigma(t))$ is the integral curve of X.

This completes the proof.

Theorem 2 : Let ϕ be a transformation of M. If a vector field X generates ϕ_t as its local 1-parameter group of transformations, then, the vector field $\phi_* X$ will generate $\phi\phi_t\phi^{-1}$ as its local 1-parameter group of transformations.

Proof : Left to the reader.

Exercise : 4. Show that a vector field X on M is invariant under a transformation ϕ on M if and only if

$$\phi \circ \phi_t = \phi_t \circ \phi$$

where ϕ_t is the local 1-parameter group of transformations induced by X.

We now give a geometrical interpretation of [X, Y], for every vector field X, Y on M.

Theorme 3 : If X generates ϕ_t as its local 1-parameter group of transformations, then, for every vector field Y on M.

$$[X, Y]_q = \lim_{t \to 0} \frac{1}{t}\left\{Y_q - ((\phi_t)_* Y)_q\right\} \text{ where } q = \phi_t(p) \text{ and } (\phi_t)_* Y_p = ((\phi_t)_* Y)\phi_t(p)$$

To prove the theorem, we require some lemmas which are stated below :

Lemma 1 : If ψ (t, p) is a function on $I_\in \times M$, where $I_\in$ is an open interval $(-\in, \in)$ such that

$$\psi\ (0, p) = 0\ , \qquad \forall\ p \in M$$

then, there exists a function h (t, p) on $I_\in \times M$ such that

$$t\ h\ (t, p) = \psi\ (t, p)$$

Moreover

$$h\ (o, p) = \psi'\ (o, p), \text{ Where } \psi' = \frac{d\psi}{dt}.$$

Proof : It is sufficient to define

$$h(t,p) = \int_0^1 \psi'(ts,p)\frac{d(ts)}{t}$$

Hence by the fundamental theorem of calculus

$$h(t,p) = \left[\frac{1}{t}\psi(ts,p)\right]_0^1$$

$$\therefore\ th(t,p) = \psi(t,p)$$

Also from above

$$h(o,p) = \int_0^1 \psi'(o,p)ds \quad = \psi'(o,p)\,[s]_0^1 = \psi'(o,p)$$

Lemma 2 : If f is a function on M and X is a vector field on M which induces a local 1-parameter group of transformations ϕ_t then there exists a function g_t defined on $I_\in \times V$, V being the neighbourhood of p of M, where

$$g_t(p) = g(t,p)$$

such that

$$f(\phi_t(p)) = f(p) + tg_t(p)$$

Moreover,

$$X_p f = g(o,p) = g_0(p)$$

Symbolically,

$$Xf = g_0 \text{ on M.}$$

Proof : Let us set

$$\tilde{f}(t,p) = f(\phi_t(p)) - f(\phi_0(p)), \quad \forall\ p \in M$$

Then $\tilde{f}(t,p)$ is a function on $I_\in \times M$ such that

$$\tilde{f}(o,p) = f(\phi_0(p)) - f(\phi_0(p)) = 0, \quad \forall\ p \in M$$

Hence by Lemma 1, there exists a function, say, g(t, p) on $I_\in \times V$, $V \subset M$ being the neighbourhood of p of M, such that

$$tg(t,p) = \tilde{f}(t,p)$$

$$\therefore \quad g(t,p) = \frac{f(\phi_t(p)) - f(\phi_0(p))}{t}$$

or, $$g(o,p) = \lim_{t \to 0} \frac{1}{t} \{f(\phi_t(p))\} - f(\phi_0(p)) = X_p f$$

As,

$$tg(t,p) = f(\phi_t(p)) - f(p)$$

we find that

$$f \circ \phi_t = f + tg_t$$

Proof of the main theorem :

Let us write

$$\phi_t(p) = q$$

$$\therefore \quad p = \phi_t^{-1}(q) = \phi_{-t}(q)$$

Now,

$$\{((\phi_t)_* Y)f\}(q) = \{Y(f \circ \phi_t)\}(p) = \{Y(f + tg_t)\}(p) \text{ by Lemma 2}$$

or $$(Yf)(q) - (((\phi_t)_* Y))(q) = (Yf)(q) - (Yf)(p) - t(Yg_t)(\phi_{-t}(q))$$

or, $$\left(\lim_{t \to 0} \frac{1}{t}\left\{Y_q - ((\phi_t)_* Y)_q\right\}\right)f = \lim_{t \to 0} \frac{(Yf)(q) - (Yf)(p)}{t} = \lim_{t \to 0}(Yg_t)(\phi_{-t}(q))$$

$$= \lim_{t \to 0} \frac{1}{t}\{(Yf)(q) - (Yf)(p)\} - (Yg_0)(q)$$

$$= \lim_{t \to 0} \frac{1}{t}\{(Yf)(q) - (Yf)(p)\} - y_q(Xf), \quad \text{by Lemma 2}$$

From the definition we find that,

$$X_q f = \lim_{t \to 0} \frac{1}{t}\{f(\phi_t(q)) - f(q)\}$$

or $$-X_q f = \lim_{t \to 0} \frac{1}{t}\{f(p) - f(q)\}$$

Taking $f = Yf$, we find from above after a few steps

$$X_q(Yf) = \lim_{t \to 0} \frac{1}{t}\{(Yf)(q) - (Yf)(p)\}$$

Thus we write,

$$\left(\lim_{t \to 0} \frac{1}{t}\left\{Y_q - ((\phi_t)_* Y)_q\right\}\right)f = X_q(Yf) - Y_q(Xf) = \{[X.Y]f\}(q), \text{ after a few steps.}$$

$$[X,Y]_q = \lim_{t \to 0} \frac{1}{t}\left\{Y_q - ((\phi_t)_* Y)_q\right\}$$

Note : We abbreviate the above result as

$$[X,Y] = \lim_{t \to 0} \frac{1}{t}\{Y - ((\phi_t)_* Y)\}$$

Corollary : 1. Show that

$$(\phi_s)_*[X,Y] = \lim_{t \to 0} \frac{1}{t}\{(\phi_s)_* Y - ((\phi_{s+t})_* Y)\}$$

Proof : From the last theorem

$$(\phi_s)_*[X,Y] = \lim_{t\to 0}\frac{1}{t}\ \left\{(\phi_s)_*Y - (\phi_s)_*(\phi_t)_*Y\right\},\ \text{ as } (\phi_s)_* \text{ is a linear mapping}$$

$$\lim_{t\to 0}\frac{1}{t}\left\{(\phi_s)_*Y - ((\phi_s\circ\phi_t)_*Y)\right\},\ \text{from a known result}$$

Using the definition of local 1-parameter group of transformations, the result follows immediately.

Corollary 2 : Show that

$$(\phi_s)_*[X.Y] = -\left(\frac{d((\phi_t)_*Y)}{dt}\right)_{t=s}$$

Proof : Left to the reader

Corollary 3 : Let X, Y generate ϕ_t and ψ_s respectively, as its local 1-parameter group of transformations. Then

$$\phi_t\circ\psi_s = \psi_s\circ\phi_t$$

if and only if [X, Y].

Proof : Let

$$\phi_t\circ\psi_s = \psi_s\circ\phi_t$$

Then from Exercise 4, the vector field Y is invariant under ϕ_t . Hence by § 1.8

$$(\phi_t)_*Y = Y$$

Consequently from Theorem 3, [X, Y] = 0

Converse result follows from corollary 2.

A vector field X on a manifold M is said to be <u>complete</u> if it induces a one parameter group of transformations on M.

Theorem 4 : Every vector field on a compact manifold M is complete.

Proof : Let X be a given vector field on M. Then by Theorem 1, X induces $\{\phi_t\}$ as its local 1-parameter group of transformations in a neighbourhood U of p of M and $t \in I_{\in} \subset R$. If p runs over M, then for each p, we get a neighbourhood U(p) and $I_{\in}$ (p), where all such U(p) from an open coverings of M. Since M is compact, every open covering {U(p)} of M has a finite subcovering $\{ U(p_i) \; : \; i = 1, ..., n \}$ say. If we let

$$\in = \min\{\in (p_1), \in (p_2), \; ..., \; \in (p_n)\}$$

then, there is a t such that for $|t| < \in$

$$\phi_t(p) : (-\in, \in) \text{ x M} \to \text{M}$$

is local 1-parameter group of transformations on M. We are left to prove that $\phi_t(p)$ is defined on R × M.

Case a) : $t \geq \in$

We write

$$t = k \cdot \frac{\in}{2} + r \quad , |r| < \frac{\in}{2}, \text{ k being integer}$$

Then $\phi_t = \phi_{k\frac{\in}{2}+r}$

$$= \phi_{k\frac{\in}{2}} \circ \phi_r$$

$$= \underbrace{\phi_{\frac{\in}{2}} \circ \phi_{\frac{\in}{2}} \; ..... \circ \phi_{\frac{\in}{2}}}_{\text{k times}} \cdot \phi_r$$

Similarly for $t \leq -\in$, we can show that

$$\phi_t = \phi_{-\frac{\in}{2}} \; .... \; \phi_{-\frac{\in}{2}} \cdot \phi_r$$

Thus ϕ_t is a local 1-parameter group of transformations on M.

Combining all the cases, we claim that ϕ_t is defined on R × M. Hence X induces ϕ_t as its 1-parameter group of transformations on a compact manifold M. Thus X is a complete vector field.

§ 1.10 Cotangent Space :

Note that χ(M) is a vector space over the field of real numbers. A mapping

$\omega : \chi(M) \to F(M)$

that satisfies

$\omega(X+Y) = \omega(X) + \omega(Y)$

$\omega(bX) = b\,\omega(X)$, $b \in$ R and for all X, Y $\in$ X(M),

is a linear mapping over R.

The linear mapping

$\omega : \chi(M) \to F(M)$

denoted by

$\omega : X \to \omega(X)$

is called a 1-form on M.

Let

$$D_1(M) = \{\omega, \mu, \ldots \mid \omega : \chi(M) \to F(M)\}$$

be the set of all 1-forms on M. Let us define

10.1) $$\begin{cases} (\omega+\mu)(X) = \omega(X) + \mu(X) \\ (b\omega)(X) \quad = b\omega(X) \end{cases}$$

It can be shown that D_1(M) is a vector space over R, called the dual of χ(M).

For every p$\in$M, $\omega(X) \in F(M)$ is a mapping

$\omega(X) : M \to R$ defined by

10.2) $$\{\omega(X)\}(p) = \omega_p(X_p)$$

so that

$$\omega_p : T_p(M) \to R$$

Thus $\omega_p \in$ dual of $T_p(M)$. We write the dual of $T_p(M)$ by $T_p^*(M)$ and is the <u>cotangent space</u> of $T_p(M)$. Elements of $T_p^*(M)$ are called the <u>covectors</u> at p of M or <u>linear functionals</u> on $T_p(M)$.

For every f$\in$ F(M), we denote the <u>total differential</u> of f by df and is defined as

10.3) $(df)_p(X_p) = (Xf)(p) = X_p f, \ \forall p$

We also write it as

10.4) (df) (X) = Xf

Exercises : 1. Show that for every f$\in$F(M), df is a 1-form on M.

2. If $(x^1, x^2, ..., x^n)$ are coordinate functions defined in a neighbourhood U of p $\in$ M, show that each $dx^i, i = 1, ..., n$ is a 1-form on U $\subset$ M.

Solution : 2 Note that

$$(dx^i)(X+Y) = (X+Y)x^i, \ (10.4) \text{ above}$$

$$= Xx^i + Yx^i$$

$$= (dx^i)(X) + (dx^i)(Y), \quad \text{by } (10.4)$$

Similarly it can be shown that

$$(dx^i)(bX) = b(dx^i)(X)$$

Thus each $dx^i, i = 1, ..., n$ is a 1-form on R.

From Exercise 2 above, it is evident that each $(dx^i)_p \in T_p^*(M)$, for i =1, ..., n. We now define

10.5) $\left(dx^i\right)_p\left(\frac{\partial}{\partial x^i}\right)_p = \delta^i_j$

Let $\omega_p \in T_p^*(M)$ be such that

10.6) $\omega_p\left(\frac{\partial}{\partial x^j}\right)_p = \left(f_j\right)_p$ where each $\left(f_j\right)_p \in R$

If possible, let $\mu_p \in T_p^*(M)$ be such that

$$\mu_p = \left(f_1\right)_p\left(dx^1\right)_p + ... + \left(f_n\right)_p\left(dx^n\right)_p$$

then

$$\mu_p\left(\frac{\partial}{\partial x^1}\right)_p = \left\{\left(f_1\right)_p (dx^1) + \cdots + (f_n)_p (dx^n)_p\right\}\left(\frac{\partial}{\partial x^1}\right)_p = (f_1)_p \text{ by (10.5)}$$

Proceeding in this manner we will find that

$$\mu_p\left(\frac{\partial}{\partial x^1}\right)_p = \left(f_1\right)_p = \omega_p\left(\frac{\partial}{\partial x^i}\right) \text{ by (10.6)}$$

As $\left\{ \frac{\partial}{\partial x^i} : i = 1,...,n \right\}$ are linearly independent, we must have

$$\mu_p = \omega_p.$$

Thus any $\omega_p \in \mathrm{T}_p^*$(M) can be expressed uniquely as

10.7) $\omega_p = \sum (f_i)_p (dx^i)_p$

$\therefore$ T_p^*(M) = span $\left\{ (dx^1)_p, ..., (dx^n)_n \right\}$

Finally if

$$\sum_i (f_i)_p (dx^i)_p = 0, \text{ then,}$$

$$\sum_i (f_i)_p \left(dx^i\right)_p \left(\frac{\partial}{\partial x^k}\right)_p = 0$$

i.e. $\left(f_k\right)_p = 0.$ by (10.5)

Similarly it can be shown that

$$\left(f_1\right)_p = ... = \left(f_n\right)_p = 0$$

Thus the set $\left(f_1\right)_p = ... = \left(f_n\right)_p = 0$ is linearly independent and we state

Theorem 1 : If $\left(x^{1,}, ...x^n\right)$ are local coordinate system in a neighbourhood U of p of M, then the linear functionals $\{(dx^i)_p := 1, ..., n\}$ on $T_p(M)$ form a basis of $T_p^*(M)$.

Note that

$(dX^i)(X) = Xx^i$ by 10.4)

$= \sum \xi^j \frac{\partial}{\partial x^j} x^i$ by 5.2) of § 1.5

10.8) $(dx^i)(X) = \xi^i$

Thus, one can find

$$(df)(X) = Xf = \sum \xi^i \frac{\partial}{\partial x^i} f = \sum \frac{\partial f}{\partial x^i} dx^i(X) \text{ from above}$$

Hence we write

10.9) $\quad df = \sum \frac{\partial f}{\partial x^i} dx^i$

For every $\omega \in D_1$ (M), we define $f\,\omega$ to be a 1 form in M and write

10.10) $\quad (f\omega)(X) = f\left(\omega(X)\right)$

Note : $D_1(M)$ is a F(M)-module

§. 1.11 r-form, Exterior Product :

An r-form is a skew-symmetric mapping

$\omega : \chi(M) \times \cdots\cdots \times \chi(M) \to F(M)$

such that

i) w is R-linear

ii) if σ is a permutation of 1,2......r with

(1, 2, ..., r)$\to \left(\sigma(1), \sigma(2), \ldots \sigma(r)\right)$ then

11.1) $\omega\left(X_{1,}\ X_{2,}\ldots\ldots, X_r\right) = \frac{1}{r!}\sum_{\sigma}(\text{sgn}\sigma)\,\omega(X_{\sigma(1)}, X_{\sigma(2)}, \ldots\ldots X_{\sigma(r)})$ where $(\text{sgn}\,\sigma)$ is + 1 or –1 according as σ is even or odd permutation .

If ω is a r-form and μ is a s-form, then, the <u>exterior product</u> or <u>wedge product</u> of ω and μ denoted by $\omega \wedge \mu$ is a (r+s)-form. defined as

11.2 $(\omega \wedge \mu)(X_1, X_2, \ldots\ldots X_r, X_{r+1,\ldots\ldots} X_s)$

$$= \frac{1}{(r+s)!}\sum_{\sigma}(\text{sgn}\,\sigma)\omega(X_{\sigma(1)}, \ldots, X_{\sigma(r)})\,\mu\,(X_{\sigma(r+1)}, \ldots\ldots\ X_{\sigma(r+s)})$$

where σ ranges over the permutation (1, 2,......r+s), $X_i \in \chi(M).i = 1, 2, \ldots\ldots, r+s$

For convenience, we write

11.3) $f \wedge g = fg,\ \ f, g \in F(M).$

It can be shown that, if ω is a r-form

11.4) $$\begin{cases} (f \wedge \omega)(X_1, \ldots, X_r) = f\omega(X_1, X_2, \ldots, X_r) \\ (\omega \wedge f)(X_1, \ldots, X_r) = f\omega(X_1,\ \ldots, X_r) \end{cases}$$

Again, if ω and μ are 1-forms, then

11.5) $(\omega \wedge \mu)(X_1, X_r) = \frac{1}{2}\left\{\omega(X_1)\mu(X_2) - \omega(X_2)\mu(X_1)\right\}$

The exterior product obeys the following properties :

11.6) $$\begin{cases} \omega \wedge \mu = -\mu \wedge \omega, \quad \omega \wedge \omega = 0 \\ f\omega \wedge \mu = f(\omega \wedge \mu) = \omega \wedge f\mu \\ f\omega \wedge g\mu = fg\omega \wedge \mu \quad , \quad \omega \wedge \mu = (-1)^{rs} \mu \wedge \omega, \quad \omega \text{ : r - form } \mu \text{ : s - form} \\ (\omega + \mu) \wedge \gamma = \omega \wedge \gamma + \mu \wedge \gamma \end{cases}$$

Exercises : 1. If ω is a 1-form and μ is a 2-form, show that

$$(\omega \wedge \mu)(X_1, X_2, X_3) = \frac{1}{3}\left\{\omega(X_1)\mu(X_2, X_3) + \omega(X_2)\mu(X_3, X_1) + \omega(X_3)\mu(X_1, X_2)\right\}$$

2. Compute

i) $(2du^1 + du^2) \wedge (du^1 - du^2)$

ii) $(6du^1 \wedge du^2 + 27du^1 \wedge du^3) \wedge (du^1 + du^2 + du^3)$

Solution : 2 i) $(2du^1 + du^2) \wedge (du^1 - du^2)$

$= 2du^1 \wedge (du^1 - du^2) + du^2 \wedge (du^1 - du^2)$

$= -2du^1 \wedge du^2 + du^2 \wedge du^1$ as $du^i \wedge du^i = 0$

$= -3\, du^1 \wedge du^2$ by 11.6)

Theorem 1 : In terms of a local coordinate system $(x^1, x^2, \ldots, x^n)$ in a neighbourhood U of p of M, an r-form ω can be expressed uniquely as

11.7) $\omega = \sum_{i_1 < i_2 < \ldots < i_r} f_{i_1 i_2 \ldots i_r}\, dx^{i_1} \wedge dx^{i_2} \ldots \wedge dx^{i_r}$ where $f_{i_1 i_2 \ldots i_r}$ are differentiable functions on M.

Proof : Let $D_r(M)$ denote the set of all differentiable r-forms on M. In terms of a local co-ordinate system $(x^1, x^2, \ldots, x^n)$ in a neighbourhood U of p of M, the set $\left\{ dx^{i_1} \wedge \ldots \wedge dx^{i_r} : 1 \le i_1 < i_2 < \ldots < i_r \le n \right\}$ form a basis of $D_r(M)$. Using 11.2) we find

i) $$(dx^{i_1} \wedge \ldots \wedge dx^{i_r})\ (X_1, X_2, \ldots, X_r) = \frac{1}{r!} \sum_{\sigma} (\operatorname{sgn} \sigma)\, dx^{i_1}\left(X_{\sigma(1)}\right) \ldots dx^{i_r}\left(X_{\sigma(r)}\right)$$

$i_1 < i_2 < \ldots < i_r$

where σ ranges over the permutation (1, 2, ..., r) and each $X_i \in \chi$ (M).

Let

ii) $$X_k = \sum_{j_m=1}^{n} \xi_k^{j_m} \frac{\partial}{\partial x^{j_m}}$$

where ξ's are functions, called the components of X_k.

Using ii), we get from i)

$$\left(dx^{i_1} \wedge \ldots\ldots \wedge dx^{i_r}\right)(X_1, \ldots, X_r) = \frac{1}{r!} \sum_{\sigma} (\operatorname{sgn}\sigma)\, dx^{i_1}\left(\sum \xi_{\sigma(1)}^{j_m} \frac{\partial}{\partial x^{jm}}\right) \ldots dx^{i_r}\left(\sum \xi_{\sigma(r)}^{j_k} \frac{\partial}{\partial x^{jk}}\right)$$
$$i_1 < i_2 < \ldots < i_r$$

Using (10.5) of § 1.10, we get from above

iii) $$(dx^{i_1} \wedge \ldots \wedge dx^{i_r})\ (X_1, X_2, \ldots, X_r) = \frac{1}{r!} \sum_{\sigma} (\operatorname{sgn}\sigma)\ \xi_{\sigma(1)}^{i_1} \ldots \xi_{\sigma(r)}^{i_r}\ \ i_1 < i_2 < \ldots < i_r$$

Using ii) in (11.1) of § 1.11, , we find

$$\omega\,(X_1, X_2 \ldots, X_r) = \frac{1}{r!} \sum_{\sigma} (\operatorname{sgn}\sigma)\ \omega\left(\sum \xi_{\sigma(1)}^{j_m} \frac{\partial}{\partial x^{j_m}}, \ldots, \sum \xi_{\sigma(r)}^{j_s} \frac{\partial}{\partial x^{j_s}}\right)$$

As each ω is R-linear, we find from above

$$= \frac{1}{r!} \sum_{\sigma} (\operatorname{sgn}\sigma) \sum_{j_m, \ldots, j_s} \xi_{\sigma(1)}^{j_m} \cdots \xi_{\sigma(r)}^{j_s}\ \omega\left(\frac{\partial}{\partial x^{j_m}}, \cdots, \frac{\partial}{\partial x^{j_s}}\right)$$

Changing the dummy indices $j_m \to i_1, \ldots, j_s \to i_r$ we get

$$= \frac{1}{r!} \sum_{\sigma} (\operatorname{sgn}\sigma) \sum_{i_1, \ldots, i_r} \xi_{\sigma(1)}^{i_1} \cdots \xi_{\sigma(r)}^{i_r}\ \omega\left(\frac{\partial}{\partial x^{i_1}}, \cdots, \frac{\partial}{\partial x^{i_1}}\right)$$

Using iii) we find from above

$$= \sum_{\substack{i_1, \ldots i_r \\ i_1 < i_2 < \ldots < i_r}} (dx^{i_1} \wedge \ldots \wedge dx^{i_r})\ (X_1, X_2 \ldots, X_r) f_{i_1 i_2 \ldots i_r}\quad , \text{ where}$$

$$\omega\left(\frac{\partial}{\partial x^{1_i}}, \cdots, \frac{\partial}{\partial x^{1_i}}\right) = f_{i_1 i_2 \ldots i_r}$$

Thus

$$\omega(X_1, X_2 ..., X_r) = \sum_{\substack{i_1,...i_r \\ i_1<i_2<...<i_r}} f_{i_1 i_2 ... i_r} (dx^{i_1} \wedge ... \wedge dx^{i_r})(X_1, ..., X_r), \quad \forall\ X_1, ..., X_r$$

Hence we can write

$$\omega = \sum_{i_1<i_2<...<i_r} f_{i_1 i_2 ... i_r} dx^{i_1} \wedge dx^{i_2} \wedge ... \wedge dx^{i_r}$$

This completes the proof.

Exercises : 3. Show that a set of 1-forms $\{\omega_1, \omega_2, ..., \omega_k\}$ is linearly dependent if and only if

$$\omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_k = 0$$

4. Let $\{\omega_1, \omega_2, ..., \omega_k\}$ be k-independent 1-forms on M. If μ_i be k 1-forms satisfying

$$\sum_i \mu_i \wedge \omega_i = 0$$

show that

$$\mu_i = \sum A_{ij}\, \omega_j \ \text{ with } \ A_{ij} = A_{ji}$$

Solution : 3. Let the given set of 1-forms be linearly dependent. Hence any one of them, say, ω_{k-1} can be expressed as a linear combination of the rest i.e.

$$\omega_{k-1} = b_1\omega_1 + b_2\omega_2 + \cdots + b_{k-2}\omega_{k-2} + b_k\omega_k, \text{ where each } b^i \in R$$

$\therefore \quad \omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_{k-1} \wedge \omega_k$

$= \omega_1 \wedge \omega_2 \wedge \cdots \wedge (b_1\omega_1 + b_2\omega_2 + ... + b_{k-2}\omega_{k-2} + b_k\omega_k) \wedge \omega_k$

$= b_1\omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_1 \wedge \omega_k + \cdots + b_k\omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_k \wedge \omega_k$

= 0 by 11.6) of this article.

Converse follows easily.

4. As $\{\omega_1, ..., \omega_k\}$ is a independent set of of 1-forms, we complete the basis of D_1 (M) by taking 1-forms $\omega_{k+1}, ..., \omega_n$. Thus the basis of D_1 (M) is given by $\{\omega_1, ..., \omega_k, \omega_{k+1}, ..., \omega_n\}$.

Consequently any 1-from μ_i, $i = 1,...k$ can be expressed as

i) $\mu_i = \sum_{m=1}^{k} A_{im}\omega_m + \sum_{p=k+1}^{n} B_{ip}\omega_p, \; i = 1, 2, \ldots k$

Given that

$$\sum_i \mu_i \wedge \omega_i = 0$$

i.e. $\mu_1 \wedge \omega_1 + \mu_2 \wedge \omega_2 + \cdots + \mu_k \wedge \omega_k = 0$

Using i) and 11.6) one gets after a few steps

$$\sum_{i<j\leq k} (A_{ij} - A_{ji})\, \omega_i \wedge \omega_j + \sum_{\substack{i\leq k \\ j>k}} B_{ji}\omega_i \wedge \omega_j = 0$$

As ω 's are given to be linearly independent, so we must have

$$A_{ij} - A_{ji} = 0 \text{ and } B_{ij} = 0$$

i.e. $A_{ij} = A_{ji}$

Consequently i) reduces to

$$\mu_i = \sum A_{ij}\omega_j \text{ with } A_{ij} = A_{ji}$$

§. 1.12. Exterior Differentiation :

The exterior derivative, denoted by d on D is defined as follows :

i) $d(D_r) \subset D_{r+1}$

ii) for $f \in D_0$, df is the total differential

iii) if $\omega \in D_r$, $\mu \in D_s$ then

$$d(\omega \wedge \mu) = d\omega \wedge \mu + (-1)^r \omega \wedge d\mu$$

iv) $d^2 = 0$

From 11.7) of § 1.11 we find that

12.1) $d\omega = \sum_{i_1<i_2<\cdots<i_r} df_{i_1\cdots i_r} \wedge dx^{i}1 \wedge \cdots \wedge dx^{i_r}$

Exercises : 1. Find the exterior differential of

i) $x^2 y\, dy - xy^2\, dx$

ii) $\cos(xy^2)\, dx \wedge dz$

iii) $x\, dy \wedge dz + y\, dz \wedge dx + z\, dx \wedge dy$

2. Find the exterior differential of

$d\omega \wedge \mu - \omega \wedge d\mu$

A form ω is said to be closed if

12.2) $d\omega = 0$

If ω is a r–form and

12.3) $d\mu = \omega$

for some (r–1) form μ then, ω is said to be an exact form.

Exercise : 3. Test whether ω is closed or not where

i) $\omega = xy\, dx + \left(\frac{1}{2}x^2 - y\right) dy$

ii) $\omega = e^x \cos y\, dx + e^x \sin y\, dy$

Theorem 1 : If ω is a 1-form, then

$$d\omega(X_1, X_2) = \frac{1}{2}\{X_1(\omega(X_2)) - X_2(\omega(X_1)) - \omega([X_1, X_2])\}$$

Proof : Without any loss of generality, one may take an 1-form as

$\omega = f\, dg\,,\ f, g \in$ F(M)

$\therefore \quad d\omega(X_1, X_2) = (df \wedge dg)(X_1, X_2)$

Using 11.5) of § 1.11, we find

$$d\omega(X_1, X_2) = \frac{1}{2}\{(df)(X_1)\, dg\,(X_2) - (df)(X_2)\,(dg)\,(X_1)\}$$

Using (10.4) of § 1.10, we get

$$d\omega(X_1, X_2) = \frac{1}{2}\{(X_1 f)(X_2 g) - (X_2 f)(X_1 g)\}$$

$$= \frac{1}{2}\{X_1(f(X_2 g) - f(X_1(X_2 g)) - X_2(f(X_1 g) + f(X_2(X_1 g))\}$$ on

using (4.6) of § 1.4

Now $\omega(X_1) = (fdg)(X_1) = f(dg(X_1))$, as $(f\omega)(X) = f(\omega(X))$

$= f(X_1 g)$ by (10.4) of § 1.10

by $\omega(X_2) = f(X_2 g)$

Thus we get from above

$$d\omega(X_1, X_2) = \frac{1}{2}\left[X_1(\omega(X_2) - X_2(\omega(X_1) - f\{X_1(X_2 g) - X_2(X_1 g)\}\right]$$

$$= \frac{1}{2}\{X_1(\omega(X_2) - X_2(\omega(X_1) - f([X_1, X_2]g)\}$$

$$\therefore \quad d\omega(X_1, X_2) = \frac{1}{2}\{ X_1(\omega(X_2)) - X_2(\omega(X_1)) - \omega([X_1, X_2]) \}$$

This completes the proof.

Existence and Uniqueness of Exterior Differentiation :

Without any loss of generality we may take an r-form as

$\omega = f_{i_1 i_2 \dots i_r} dx^{i_1} \wedge \dots \wedge dx^{i_r}$, $f_{i_1 \dots i_r} \in$ F(M)

Let us define an R-linear map

d : D $\to$ D as

12.4) $d\omega = df_{i_1 i_2 \dots i_r} \; dx^{i_1} \wedge \dots \wedge dx^{i_r}$

Clearly i) d $(D_r) \subset D_{r+1}$ and

ii) if ω is a 0-form, then $d\omega$ is the total differential of ω.

iii) Let $\mu \in D_s$ and it is enough to consider

$\mu = g_{j_1 \dots j_s}\, dx^{j_1} \wedge \dots \wedge dx^{j_s}$, $g_{j_1 \dots j_s} \in$ F(M)

then $d(\omega \wedge \mu) = d\left(f_{i_1 i_2 \dots i_r}\, g_{j_1 \dots j_s}\, dx^{i_1} \wedge \dots \wedge dx^{i_r} \wedge dx^{j_1} \wedge \dots \wedge dx^{j_s}\right)$

Using 12.1 we get

$$d(\omega \wedge \mu) = d(f_{i_1 \dots i_s}\, g_{j_1 \dots j_s}) \wedge dx^{l_1} \wedge dx^{j_i} \wedge \dots \dots \wedge dx^{j_s}$$

$$= (g_{j_1 \dots j_s}\, df_{i_1 \dots i_r} + f_{i_1 \dots i_r}\, dg_{j_1 \dots j_s}) \wedge dx^{i_r} \wedge \dots \wedge dx^{j_1} \wedge dx^{ji} \wedge \dots \wedge dx^{j_s}$$

$$= g_{j_1 \dots j_s}\, df_{i_1 \dots i_r} \wedge dx^{i_r} \wedge dx^{j_1} \wedge \dots \wedge dx^{j_s} + f_{i_1 \dots i_r}\, dg_{j_1 \dots j_s} \wedge dx^{i_1} \wedge \dots \wedge dx^{i_r} \wedge dx^{j_1} \wedge \dots \wedge dx^{j_s}$$

$$= df_{i_1 \dots i_r} \wedge dx^{i_1} \wedge dx^{i_r} \wedge g_{j_i \dots j_s} dx^{j_1} \wedge \dots \wedge dx^{j_s} + (-1)^r f_{i_1 \dots i_r} dx^{i_1} \wedge \dots \dots dx^{i_r} \wedge dg_{j_1 \dots j_s} \wedge dx^{j_i} \wedge dx^{j_s}$$

$$= d\omega \wedge \mu + (-1)^r \omega \wedge d\mu$$

iv) Again using (10.9) of § 1.10 in (12.4) we see that

$$d\omega = \sum_{i_k} \frac{\partial f}{\partial x^{i_k}} dx^{i_k} \wedge dx^{i_1} \wedge \dots \wedge dx^{i_r}$$

or d $d(d\omega) = \sum_{i_s} \sum_{i_k} \frac{\partial^2 f}{\partial x^{i_s} \partial x^{i_k}} dx^{i_s} \wedge dx^{i_k} \wedge dx^{i_1} \wedge \dots \wedge dx^{i_r}$

$= 0,$

If $i_s \neq i_k$, then, $dx^{i_s} \wedge dx^{i_k} = 0$

Thus existence of such d is established.

It is easy to establish the uniqueness of d.

Thus there exist a unique exterior differentiation on D.

§. 1.13 Pull-back Differential Form :

Let M be an n-dimensional and N be an m-dimensional manifold and

$f : M \to N$

be a differentiable mapping. Let $T_p(M)$ be the tangent space at p of M where $T^*_{f(p)}(N)$ is its dual. Let $T_{f(p)}(N)$ be the tangent space at f(p) of N where $T^*_{f(p)}(N)$ is its dual. If $(x^1,....x^n)$ and $(y^1,....y^m)$ are the local corrdinate system at p of M and at f(p) of N respectively, then, it is known that $\{\frac{\partial}{\partial x^i} : i=1,.....,n\}$ *and* $\{\frac{\partial}{\partial y^j} : j=1,.....,m\}$ are respectively the basis of $T_p(M)$ and $T_{f(p)}(N)$. Consequently $\{dx^i : i=1,...n\}$ and $\{dy^j : j = 1, ..., m\}$ are the basis of $T_p^*(M)$ and $T^*_{f(p)}(N)$ respectively.

Let ω be a 1-form on N. We define an 1-form on M, called the pull-back 1 form of ω on M, denoted by $f^*\omega$, as follows

13.1) $\{f^*(\omega_{f(p)})\}(X_p) = \left(f^*\omega\right)_p (X_p) = \omega_{f(p)}(f_*(X_p), \forall$ p of M.

where f_*, $f^*$are already defined in § 1.7

So, we write

13.2) $f^*(\omega_{f(p)}) = (f^*\omega)_p$

then, by 7.4) of § 1.7, we get from 13.1, on using 13.2)

13.3) $(f^*\omega)_p(X_p) = \omega_{f(p)}(f_*X)_{f(p),} \ \forall\, p \ of \ M$

Therefore we may write, for a 1 form ω on N and a vector field X on M by

13.4) $(f^*\omega)(X) = \omega\,(f_*X)$

Theorem 1 : If f is a mapping from an n-dimensional manifold M to an m-dimensional manifold N, where $(x^1, x^2,....x^n)$ is the local coordimate system in a neighbourhood of a point p of M and $(y^1....y^m)$ is the local coordinate system in a neighbourhood of f(p) of N, then

$$f^*(dy^i)_{f(p)} = \sum_{i=1}^{n}(\frac{\partial f^j}{\partial x^i})_p \,(dx^i)_p \qquad \text{where} \quad f^j = y^j.f\ ,\ i = 1,....m$$

Proof : Since $f^*(dy^i)_{f(p)}$ is a co-vector at P on M, it can be expressed as the linear combination of the basis co-vectors $(dx^i)_p$ at P and we take

$$f^*(dy^j)_{f(p)} = \sum_{i=1}^{n} a_i^j (dx^i)_p$$

Where a_i^j 's are unknown s to be determined

or $\{f^*(dy^j)_{f(p)}\}\left(\frac{\partial}{\partial x^k}\right)p = \sum_i a_i^j (dx^i)_p \left(\frac{\partial}{\partial x^k}\right)p$

usinng 10.5 of § 1.10 we find that

$$\left(f^*(dy^i)_{f(p)}\right)\left(\frac{\partial}{\partial x^k}\right)_p = a_i^j\ \delta_k^i = a_k^j \text{ for } (dx^i)_p\left(\frac{\partial}{\partial x^k}\right)_p = \frac{\partial x^i}{\partial k^k} = \delta_k^i$$

By (13.1). one reduces to

$$\left(dy^j\right)_{f(p)}\left\{f_*\left(\frac{\partial}{\partial x^k}\right)_p\right\} = a_k^j$$

using Theorem 1 of § 1.7 we find

$$\sum_{j=i}^{m}\left(dy^j\right)_{f(p)}\left(\frac{\partial f^s}{\partial x^k}\right)_p\left(\frac{\partial}{\partial y^s}\right)_{f(p)} = a_k^j$$

Using (10.5) of § (1.10) we find

$$\left(\frac{\partial f^j}{\partial x^k}\right)_p = a_k^j$$

Thus we get

$$f^*(dy^j)_{f(p)} = \sum_{i=1}^{n}\left(\frac{\partial f^j}{\partial x^k}\right)_p \left(dx^i\right)_p \qquad , \ j = 1, \dots, m; \ \ f^j = y^j \circ f$$

Note : 1. Using (10.9) of § 1.10, one find from above theorem

13.5) $f^*(dy^j)_{f(p)} = (df^j)_p, j = 1, \dots. m$

we can also write it as

13.6) $f^*(dy^j)_{f(p)} = df^j = d\left(y^j . f\right)$

2. If ω is a 1-form, then, its pull-back 1-form $f^*\omega$ is given by

13.7) $f^*\omega = \sum_j \omega_j df^j$, where ω_j are the components of ω

(Prove it.)

Exercises : 1 If $f: M \to R^3$ be such that

$f(u,\nu) = (u\cos\nu, u\sin\nu, a\nu)$ where

$x^1 = u\cos\nu, x^2 = u\sin\nu, x^3 = a\nu$

then for a given 1-form ω , $\omega = x^1dx^1 - dx^2 + x^2dx^3 on\ R^{3,}$ compute $f^*\omega$.

2. If $f: M \to R^3$ be such that

$f(u,\nu) = (a\cos u\ Sin\nu, a\ Sinu\ Sin\nu, a\ Cos\nu)$ then for a given 1-form ω

$\omega = dx^1 + adx^2 + dx^3$ on R^3, determine $f^*\omega$.

3. Let ω be the 1-form in $R^2 - \{o,o\}$ by

$$\omega = -\frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy.$$

Let U be the set in the plane (r,θ) given by

$U = \{r > 0 ; 0 < \theta < 2\pi\}$

and let f : U$\to R^2$ be the map $f(r,\theta) = \begin{cases} x = r\ Cos\theta, \\ y = r\ Sin\theta \end{cases}$ compute $f^*\omega$

Let us now suppose that ω be a r-form on N. In the same manner, as defined earlier, we define an r-form on M, called the pull-back r-form on M, denoted by $f^*\omega$, as follows :

13.8) $\left(f^*\left(\omega_{f(p)}\right)\right)\left((X_1)_p, \ldots, (X_r)_p\right) = \omega_{f(p)}\left(f_*(X_1)_p, \ldots, f_*(X_r)_p\right), \forall p$

We also write it as

13.9) $(f^*\omega)(X_1 \dots, X_r) = \omega(f_* X_1, \dots, f_* X_r)$

Proposition : 1. Let

$f : M^n \to N^m$

be a map, ω and μ be r-forms on N and g be a 0-form on N. Then

a) $f^*(\omega + \mu) = f^*\omega + f^*\mu$

b) $f^*(g\omega) = f^*(g) f^*\omega$

Proof : a) As ω and μ are r-forms on N, $(\omega + \mu)$ is also so. Hence

$\left(f^*(\omega+\mu)_{f(p)}\right)(X_1, X_2 \dots, X_r) = (\omega+\mu)_{f(p)}(f_* X_1, \dots, f_* X_r)$

$= \omega_{f(p)}(f_* X_1, \dots, f_* X_r) + \mu_{f(p)}(f_* X_1, \dots, f_* X_r)$

$= \left(f^*\left(\omega_{f(p)}\right)\right)(X_1, \dots, X_r) + \left(f^*\left(\mu_{f(p)}\right)\right)(X_1, \dots, X_r)$ by 13.8)

$\therefore\ f^*(\omega+\mu)_{f(p)} = f^*(\omega)_{f(p)} + f^*(\mu)_{f(p)}, \quad \forall f(p)$

Hence

$f^*(\omega + \mu) = f^*\omega + f^*\mu$

b) Note that if ω is a r-form and g is a o-form, then $g\omega$ is again a r-form. Using (13.8) one gets

$\left(f^*(g\omega)_{f(p)}\right)(X_1, \dots, X_r) = (g\omega)_{f(p)}(f_* X_1, f_2 X_2, \dots, f_* X_r)$

$= \left(g(fp)\omega_{f(p)}\right)(f_* X_1, f_* X_2, \dots, f_* X_r)$

$= \left((g \circ f)(p)\omega_{f(p)}\right)(f_* X_1, f_* X_2, \dots, f_* X_r)$

$= (g \circ f)(p)\omega_{f(p)}(f_* X_1, \dots, f_* X_r)$

$= \left(f^*(g)(p)(f_* \omega_{f(p)}\right)(f_* X_1, \dots, f_* X_r)$

or $\left(f^*(g\omega)_{f(p)}\right) = f^*(g)(p)\left(f^*(\omega)_{f(p)}\right)$

or $\left(f^*(g\omega)\right)_p = \left(f^*(g)\right)(p)\left(f^*(\omega)\right)_p$, $\forall$ p

Hence $f^*(g\omega) = f^*(g)f^*(\omega)$.

Exercises : 4. Show that

$$f^*(\omega \wedge \mu) = f^*\omega \wedge f^*\mu$$

5. Prove that

$$(f \circ h)^*\omega = h^*(f^*\omega)$$

Note : From Theorem 1 of § 1.11, we see that, any r-form ω can be expressed as

$$\omega = \sum_{i_1<i_2<\ldots<i_r} g_{i_1 i_2 \ldots i_r}\, dx^{i_1} \wedge \ldots \wedge dx^{i_r}$$

where $g_{i_1 i_2 \ldots i_r}$ are differentiable functions on N. Then

$$f^*\omega = \sum_{i_1<i_2<\ldots<i_r} f^*\left(g_{i_1 i_2 \ldots i_r}\, dx^{i_1} \wedge \ldots \wedge dx^{i_r}\right)$$

$= \sum f^*\, g_{i_1 i_2 \ldots i_r}\, f^*dx^{i_1} \wedge \ldots \wedge f^*dx^{i_r}$ by the Proposition 1(b) and Exercise 4 above

$$= \sum \left(g_{i_1 \ldots i_r} \circ f\right) f^*dx^{i_1} \wedge \ldots \wedge f^*dx^{i_r}$$

Using 13.5) of § 1.13 we see that

13.10) $$f^*\omega = \sum_{i_1<i_2<\ldots<i_r} \left(g_{i_1 \ldots i_r} \circ f\right) df^{i_1} \wedge \ldots \wedge df^{i_r}$$

Exercise : 7. Let M be a circle and M' be R^2 so that

$$f : M \to M'$$

be defined by

$$x^1 = r\cos\theta\,,\ x^2 = r\sin\theta$$

If $\omega = a\,dx^1 + b dx^2$ and $\mu = \frac{1}{a}dx^1 + \frac{1}{b}dx^2$, find $f^*(\omega \wedge \mu)$

Solution : In this case,

$$\omega_1 = a,\ \omega_2 = b,\ \mu_1 = \frac{1}{a},\ \mu_2 = \frac{1}{b}$$

$$df^1 = \cos\theta dr - r\sin\theta\, d\theta$$

$$df^2 = \sin\theta\, dr + r\cos\theta\, d\theta$$

$\therefore \quad f^*\omega = a(\cos\theta\, dr - r\sin\theta\, d\theta) + b(\sin\theta\, dr + r\cos\theta\, d\theta)$

$= (a\cos\theta + b\sin\theta)\, dr + (br\cos\theta - ar\, sin\theta)\, d\theta$

and $f^*\mu = \frac{1}{a}(\cos\theta\, dr - r\sin\theta\, d\theta) + \frac{1}{b}(\sin\theta\, dr + r\cos\theta\, d\theta)$

$= \left(\frac{1}{a}\cos\theta + \frac{1}{b}\sin\theta\right)dr + \left(\frac{r}{b}\cos\theta - \frac{r}{a}\sin\theta\right)d\theta$

Using Exercise 5, one finds that

$f^*(\omega \wedge \mu) = f^*\omega \wedge f^*\mu$

$= \{(a\cos\theta + b\, Sin\theta)dr + (b\, r\cos\theta - a\, r\sin\theta)\, d\theta\}$

$\wedge\left\{\left(\frac{1}{a}\cos\theta + \frac{1}{b}\sin\theta\right)dr + \left(\frac{r}{b}\cos\theta - \frac{r}{a}\sin\theta\right)d\theta\right\}$

$= (a\cos\theta + b\sin\theta)\left(\frac{r}{b}\cos\theta - \frac{r}{a}\sin\theta\right)\, dr \wedge d\theta +$

$+(b\, r\cos\theta - a\, r\sin\theta)\left(\frac{1}{a}\cos\theta + \frac{1}{b}\sin\theta\right)\, d\theta \wedge dr$

$= r\left(\frac{a}{b} - \frac{b}{a}\right) dr \wedge d\theta$ where $d\theta \wedge dr = -dr \wedge d\theta$.

Theorem 2 : For any form ω,

$$d(f^*\omega) = f^*(d\omega)$$

where the symbols have their usual meanings.

Proof : We shall consider the following cases.

i) ω is a o-form

ii) ω is a r-form

Case i) : In this case, let $\omega = h,$ where h is a differentiable function

Then $\quad \left\{f^*(dh)\right\}(X) = dh(f_* X)$

$= (f_* X)h$ by (10.4) of § 1.10

$= X(h \circ f)$ by (7.3) of § 1.7

$= d(h \circ f)(X)$ by (10.4) of § 1.10

$= \left\{d(f^* h)\right\}(X)$ by (10.4) of § 1.10

or $\quad f^*(dh) = d(f^* h)$

The result is true in this case.

Case ii) : In this case, we assume that the result is true for $(r-1)$ form. Without any loss of generality, we may take an r-form ω as

$$\omega = g_{i_1 i_2 \dots i_r} dx^{i_1} \wedge \dots \wedge dx^{i_r}$$

$$\text{or } f^*\omega = f^*\left(g_{i_1 \dots i_r}\, dx^{i_1} \wedge \dots \wedge dx^{i_r}\right)$$

$$= f^*\left(g_{i_1 \dots i_r}\, dx^{i_1} \wedge \dots \wedge dx^{i_r}\right)$$

$$= f^*\left(g_{i_1 \dots i_r}\, dx^{i_1} \wedge \dots \wedge dx^{i_{r-1}}\right) \wedge f^*(dx^{i_r})$$

or $\quad d(f^*\omega) = d\left\{f^*\left(g_{i_1 \dots i_r}\, dx^{i_1} \wedge \dots \wedge dx^{i_{r-1}}\right) \wedge f^*(dx^{i_r})\right\}$

Using (12.1) of § 1.12 we find that

$$d(f^*\omega) = d\left\{f^*\left(g_{i_1 \dots i_r}\, dx^{i_1} \wedge \dots \wedge dx^{i_{r-1}}\right) \wedge f^*(dx^{i_r})\right\} +$$

$$+(-1)^{r-1} f^*\left(g_{i_1 \dots i_r}\, dx^{i_1} \wedge \dots \wedge dx^{i_{r-1}}\right) \wedge d\left(f^*(dx^{i_r})\right)$$

Note that dx^{i_r} is a 1-form and hence the theorem is true in this case. Thus

$$d\left(f^*(dx^{i_r})\right) = f^*\left(d(dx^{i_r})\right) = 0 \text{ by (12.1) of § 1.12}$$

Hence

$$d(f^*\omega) = d\left\{f^*\left(g_{i_1 \dots i_r}\, dx^{i_1} \wedge \dots \wedge dx^{i_{r-1}}\right)\right\} \wedge f^*(dx^{i_r})$$

$$= f^*\left\{d\left(g_{i_1 \dots i_r}\, dx^{i_1} \wedge \dots \wedge dx^{i_{r-1}}\right)\right\} \wedge f^*(dx^{i_r}) \text{ , as}$$

the result is true for $(r-1)$ form

$$= f^*\left\{\left(dg_{i_1 \dots i_r} \wedge dx^{i_1} \wedge \dots \wedge dx^{i_{r-1}}\right)\right\} \wedge f^*(dx^{i_r}) \text{ by (12.1) of § 1.12}$$

$$= f^*\left(dg_{i_1 \dots i_r} \wedge dx^{i_1} \wedge \dots \wedge dx^{i_{r-1}} \wedge dx^{i_r}\right) \text{ by known result}$$

Thus $\quad d(f^*\omega) = f^*(d\omega)$

and hence the result is true for r-form also.

Combining we claim that

$$d(f^*\omega) = f^*(d\omega)$$

i.e. d and f commute each other.

REFERENCES

1. W.M.Boothby : An Introduction to Differentiable Manifolds and Riemannian Geometry.
2. Kobayashi & Nomizu : Foundations of Differentiable Geometry, Volume I
3. N. J. Hicks : Differentiable Manifold
4. Y. Matsushima : Differentiable Manifold

UNIT - 2

§. 2.1 Lie group, Left translation, Right translation :

Let G be a differentiable manifold. If G is a group and if the map

$$(g_1, g_2) \rightarrow g_1 g_2$$

from G $\times$ G to G and the map

$$g \rightarrow g^{-1}$$

from G to G are both differentiable, then G is called a Lie group.

Exmaple : Let GL(n, R) denote the set of all nonsingular n $\times$ n matrices over real numbers. GL(n, R) is a group under matrix multiplication. Define

$$\phi\,(A) = (a_{11}, a_{12}, \ldots, a_{1n}\,; a_{21}, a_{22}, \ldots, a_{2n}\,; \ldots; a_{n1}, a_{n2}, \ldots, a_{nn})$$

then

$$\phi : GL(n, R) \rightarrow R^{n^2}$$

is a mapping of class C^{∞}. Hence GL(n, R) is a Lie group.

Note : Lie groups are the fundamental building blocks for gauge theories.

For every a $\in$ G, a mapping

$$L_a : G \rightarrow G$$

defined by

2.1) $\quad L_a x = ax, \quad \forall\, x \in G$

is called a **Left translation** on G.

Similarly, a mapping

$$R_a : G \rightarrow G$$

defined by

2.2) $\quad R_a x = xa, \quad \forall\, x \in G$

is called a **right translation** on G.

Note that

$$L_a L_b x = L_a(bx) = abx \text{ and } L_{ab} x = abx$$

$$\therefore\ L_a\, L_a = L_{ab}$$

$$R_a R_b x = R_a(bx) = xba \text{ and } R_{ab} x = xba$$

$$\therefore\ R_a R_b = R_{ba}$$

$$L_a R_b x = L_a(xb) = axb \text{ and } R_b L_a x = R_b(ax) = axb$$

$$\therefore\ L_a R_b = R_b L_a$$

Thus

2.3) $L_a L_b = L_{ab}, \ R_a R_b = R_{ba}, \ L_a R_b = R_b L_a$

Again

$L_b L_a x = L_b(ax) = bax \neq abx \neq L_a L_b x$, Thus

2.4) $L_b L_a \neq L_a L_b$, unless G is commutative

Taking $b = a^{-1}$ in 2.3) we find

$$L_a L_{a^{-1}} = L_{aa^{-1}} \text{ by 2.3)}$$

$$= L_e$$

Thus

2.5) $L_{a^{-1}} = (L_a)^{-1}$

It is evident that, for every a $\in$ G, each L_a and R_a are diffeomorphism on G.

Exercise : 1 Show that the set of all left (right) translation on G form a group.

2. Let $\phi : G_1 \to G_2$ be a homeomorphism of a Lie group G_1 to another Lie group G_2. Show that

i) $\phi \circ L_a = L_{\phi(a)} \circ \phi$

ii) $\phi \circ L_b = R_{\phi(b)} \circ \phi, \quad \forall\, a, b$ in G.

3. Let ϕ be *a* 1-1 non-identity map from G to G. If

$$\phi \circ L_g = L_g \circ \phi$$

is satisfied for all g $\in$ G, then there is a h $\in$ G such that $\phi = R_h$.

Solution : 2. From the definition of group homeomorphism of a Lie group G_1 to another Lie group G_2,

$$\phi(ab) = \phi(a)\phi(b), \quad \forall\, a, b \text{ in } G_1$$

i) $(\phi \circ L_a)x = \phi(L_a x) = \phi(ax) = \phi(a)\phi(x) = L_{\phi(a)}\,\phi(x) = (L_{\phi(a)} \circ \phi)x$, $\forall\, x$ in G_1

$\therefore\ \phi \circ L_a = L_{\phi(a)} \circ \phi$

Similarly ii) can be proved.

3. As G is a group, e $\in$ G (identity). Further ϕ is a 1–1 map from G to G, so for e $\in$ G, there is h in G such that

$$\phi(e) = h$$

Note that

$\phi(e) \neq e$, because, ϕ is not an identity map.

Now for g $\in$ G,

$$g = ge$$

$$\begin{aligned}
\therefore\ \phi(g) &= \phi(ge) \\
&= \phi(L_g e) \\
&= (\phi \circ L_g)(e) \\
&= (L_g \circ \phi)(e), \text{ as given} \\
&= Lg(\phi(e)) \\
&= Lgh \\
&= gh \\
&= R_h g
\end{aligned}$$

$\therefore\ \phi = R_h$, $\forall\ g$

§. 2.2. Invariant Vector Field :

We have already defined a vector field to be invariant under a transformation in § 1.8. Note that, in a Lie group G, for every *a*, b in G, each L_a , R_b is a transformation on G. Thus we can define invariant vector field under L_a , R_b.

A vector field X on a Lie group G is called a **left invariant** vector field on G if

2.6) $(L_a)_* X_p = X_{L_a(p)}$, $\forall$ p $\in$ G , where $(La)_*$ is the differential of L_a.

Thus from § 1.7

$$\left((L_a)_* X_p\right)_{L_a(p)} = X_{L_a(p)}$$

We write it as

2.7) $(L_a)_* X = X$

Similarly for a right invariant vector field, write

2.8) $(R_a)_* X = X$

From § 1.7) we know that

$$\left((L_a)_* X_p\right)g = X_p(g \circ L_a)$$

or $\left((L_a)_* X_p\right)_{L_a(p)} g = X_p(g \circ L_a)$

If $L_a(p) = q$ then $p = (L_a)^{-1}q = L_{a^{-1}}q = a^{-1}q$

Thus the above relation reduces to

2.9) $\left((L_a)_* X\right)_q g = X_{a^{-1}q}\,(g \circ L_a)$

Let g be the set of all left invariant vector field on G.

If X, Y, $\in$ g, a, b $\in$ R, then

2.10) $(L_p)_*(aX + bY) = a(L_p)_* X + b(L_p)_* Y = aX + bY$, $(L_p)_*$ being linear explained in Unit 1.

2.11) $(L_p)_*[X,Y] = \left[(L_a)_* X, (L_p)_* Y\right]$, see § 1.7 = [X, Y]

Thus $aX + bY \in g$ and $[X,Y] \in g$. Consequently g is a vector space over R and also a Lie-algebra. The Lie algebra formed by the set of all left invariant vector fields on G is called the **Lie algebra of the Lie group G.**

Note that every left invariant vector field is a vector field i.e.

$$g \subset \chi(G)$$

where $\chi(G)$ denotes the set of all vector field on G. The converse is not necessarily true.

The converse will be true if a condition is satisfied by a vector field. The following theorem states such condition.

Theorem 1 : A vector field X on a Lie group G is left invariant if and only if for every $f \in F(G)$

2.12) $$(Xf) \circ L_a = X(f \circ L_a)$$

Proof : Let X be a left invariant vector field on a Lie group G. Then for every $f \in F(G)$, we have from (2.6)

$$\left\{(L_a)_* X_p\right\} f = X_{L_a(p)} f$$

or $$X_p(f \circ L_a) = (Xf) L_a(p) \quad \text{by Q 1.7}$$

or $$\left\{X(f \circ L_a)\right\}(p) = (Xf \circ L_a)(p) \quad , \ \forall p \in G$$

$$\therefore \ Xf \circ L_a = X\,(f \circ L_a)$$

Conversely let (2.12) be true. Reversing the steps one gets the desired result.

Note : i) The behaviour of a Lie group is determined largely by its behaviour in the neighbourhood of the identity element e of G. The behaviour can be represented by an algebraic structure on the tangent space of e, called the **Lie algebra** of the group.

ii) Note that, two vector spaces U and V are said to be isomorphic, if a mapping

$$f : \mathrm{U} \to \mathrm{V}$$

is i) linear and ii) has an inverse $f^{-1} : \mathrm{V} \to \mathrm{U}$

Theorem 2 : As a vector space, the Lie subalgebra g of the Lie group G is isomorphic to the tangent space $T_e(G)$ at the identity element $e \in G$.

Proof : Let us define a mapping

$\phi : g \to T_e$ (G) by

i) $\phi(X) = X_e$

Note that, for every X, Y in g, $X + Y \in g$ and

$\phi(X+Y) = (X+Y)_e$ by i)

$= X_e + Y_e$

$= \phi(X) + \phi(Y)$

Also for $b \in R$, $bX \in g$ and

$\phi(bX) = (bX)_e$ by i)

$= bX_e$

$= bX$ by i)

Thus ϕ is linear.

We choose $X_a \in T_a(G)$ such that

ii) $(L_a)_* V_e = X_a$, , Where $V_e \in T_e(G)$.

Then $(L_s)_* X_{s^{-1}a} = (L_s)_*(L_{s^{-1}a})_* \mathrm{V}_e$ from above

$= \left(L_s \circ L_{s^{-1}a}\right)_* \mathrm{V}_e$ from § 1.7

$= \left(L_{ss^{-1}} a\right)_* \mathrm{V}_e$ by (2.3)

$= (L_a)_* \mathrm{V}_e$

$= X_a$, as chosen

or $\left((L_s)_* X\right)_{L_s(s^{-1}a)} = X_{L_s(s^{-1}a)}$ by Q 1.7

or $(L_s)_* X = X$

$\therefore$ $X \in g$

We define

$\phi^{-1} : \mathrm{T_e(G)} \to g$ by

iii) $\phi^{-1}(\mathrm{V_e}) = X$

Then $(\phi\phi^{-1})\mathrm{V_e} = \phi\big(\phi^{-1}(\mathrm{V_e})\big) = \phi(\mathrm{X}) = \mathrm{X_e}$ ii), where $(\mathrm{L_e})_*$ is the identity differential on G.

or $\quad (\phi\phi^{-1})\mathrm{V_e} = \mathrm{V_e}$

Further, $\quad (\phi^{-1}\phi)X = \phi^{-1}\big(\phi(X)\big) = \phi^{-1}(X_e)$, by i)

$= \phi^{-1}\big((\mathrm{L_e})_*\mathrm{V_e}\big)$ by ii)

$= \phi^{-1}(\mathrm{V_e})$

$= X$ by iii)

Thus an inverse mapping exists and we claim that

$$g \cong \mathrm{T_e(G)}$$

Exercises : 1. If, X, Y are left invariant vector fields, show that [X, Y] is also so.

2. If c_{ij}^{k} $(i, j, k = 1, 2, \ldots, n)$ are structure constants on a Lie group G with respect to the basis $\{X_1, X_2, \ldots, X_n\}$ of g, show that

i) $c_{ij}^{k} = -c_{ji}^{k}$

ii) $c_{ij}^{k}\, c_{ks}^{t} + c_{js}^{k}\, c_{ki}^{t} + c_{si}^{k}\, c_{kj}^{t} = 0$

Solution : 1. From Q 1.7), we see that

$\{(L_a)_*[X, Y]\}f = [X, Y](f \circ L_a)$

$= X\big(Y(f \circ L_a)\big) - Y\big(X(f \circ L_a)\big)$, from the definition of Lie Bracket

$= X\{((L_a)_*Y)f\} - Y\{((L_a)_*X)f\}$ by § 1.7

$= X(Yf) - Y(Xf)$ by (2.7)

$= [X, Y]f$ from the definition of Lie Bracket

$\therefore \; (L_a)_*[X, Y] = [X, Y], \forall f$

Using (2.7), we see that [X, Y] is a left invariant vector field.

2. Using problem 1 above, we see that every $[X_i, X_j] \in g$ as $X_i \in g$, $i = 1, \ldots, n.$

Since $\{X_1, X_2, \ldots, X_n\}$ is a basis of g, every $[X_i, X_j] \in g$ can be expressed uniquely as,

1) $[X_i, X_j] = c_{ij}^k \; X_k$ where $c_{ij}^k \in$ R

i) Note that if $i = j, [X_i, X_j] = 0$

So, let $i \neq j.$ Then from a known result,

$[X_i, X_j] = -[X_j, X_i]$

Using 1) we find that

$c_{ij}^k \; X_k = -c_{ji}^k \; X_k$

As the set $\{X_1, \ldots, X_n\}$ is a basis of g and hence linearly independent, we must have

$c_{ij}^k = -c_{ji}^k$

ii) Using Jacobi Identity, we find that

$\big[[X_i, X_j], X_s\big] + \big[[X_j, X_s], X_i\big] + \big[[X_s, X_i], X_j\big] = \theta$

Hence from 1)

$c_{ij}^k \; [X_k, X_s] + c_{js}^k \; [X_k, X_i] + c_{si}^k \; [X_k, X_j] = \theta$ as $[bX, Y] = b[X, Y], b \in R$

Again applying 1) , we find that

$c_{ij}^k \; c_{ks}^t \; X_t + c_{js}^k \; c_{ki}^t \; X_t + c_{si}^k \; c_{kj}^t \; X_t = \theta$

As $\{X_1, \ldots, X_n\}$ is a basis and hence linearly independent, we must have

$c_{ij}^k \; c_{ks}^t + c_{js}^k \; c_{ki}^t + c_{si}^k \; c_{kj}^t = \theta$

§. 2.3 Invariant Differential Form :

A differential form ω on a Lie group G is said to be left invariant if

2.13) $L_a^*\left(\omega_{L_a(p)}\right) = \omega_p, \quad \forall\ p \in \mathrm{G}$

we write it as

2.14) $L_a^*\ \omega = \omega$ and call $L_a^*\ \omega$, the pull-back differential form of ω.

Similarly, a differential form ω on a Lie group G is said to be right invariant if

2.15) $R_a^*\ \omega = \omega$

A differential form, which is both left and right invariant, is called a biinvariant differential form.

Exercises : 1. If ω_1, ω_2 are left invariant differential forms, show that, each $d\omega, \omega_1 \wedge \omega_2$ is also so.

2. Prove that a differential 1-form ω on a Lie group is left invariant if and only if for every left invariant vector field X on G, ω(X) is a constant function on G.

3. Let ϕ : G $\to$ G be such that $\phi(a) = a^{-1}$, $\forall\ a \in G$. Show that a form ω is left invariant if and only if $\phi^*\omega$ is right invariant.

4. Prove that the set of all left invariant forms on G is an algebra over R. Such a set is denoted by A, say.

5. If g^* denotes the dual space of g, then, prove that

$$\mathrm{A} \cong g^*$$

where A is the set already defined in Exercise 4 above.

Solution : 1. From Q 1.13, we see that

$$L_a^*\ (d\omega_1) = d\left(L_a^*\ \omega_1\right)$$

where $L_a^*\ \omega_1$ is the pull-back 1 form of ω_1

Using on (2.14) on the right hand side of the above equation, we see that

$$L_a^*\ (d\omega_1) = d\omega_1$$

Consequently, $d\omega_1$ is a left invariant differential form.

It can be proved easily that $\omega_1 \wedge \omega_2$ is a left invariant differential form.

2. Let us consider a differential 1-form ω. Then for every $a \in G$, $L_a^* \omega$ will be defined as the pull-back differential 1-form. Consequently from the definition of pull-back.

$$\left(L_a^* \omega_{L_a(p)}\right)(X_p) = \omega_{L_a(p)}\left((L_a)_* X_p\right), \ \forall \ p \in \mathrm{G}$$

Let us consider X to be left invariant. Then on using (2.6) on the right hand side of the above equation, we get

1) $$\left(L_a^* \omega_{L_a(p)}\right)(X_p) = \omega_{L_a(p)}\left(X_{L_a(p)}\right)$$

Let us now consider ω to be left invariant 1-form. Then by (2.13), we get from 1)

$$\omega_p(X_p) = \omega_{L_a(p)}\left(X_{L_a(p)}\right)$$

$$= \omega_{ap}(X_{ap})$$

Taking $p = e$, we see that

$$\omega_e(X_e) = \omega_{ae}(X_{ae}) = \omega_a(X_a)$$

Consequently, ω(X) is a constant function on G.

Conversely, if ω(X) is a constant function on G, then

$$\omega_p(X_p) = \omega_{ap}(X_{ap})$$

Hence 1) reduces to

$$\left(L_a^* \omega_{L_a(p)}\right) X_p = \omega_p(X_p)$$

or $\quad L_a^* \omega_{L_a(p)} = \omega_p \quad$ which is (2.13)

Thus ω is a left invariant differential form.

This completes the proof.

Theorem 1 : If g is a Lie subalgebra of a Lie group G and g^* denotes the set of all left invariant form on G, then

$$d\omega(X,Y) = -\frac{1}{2}\,\omega([X,Y]) \text{ where } \omega \in g^*, \quad X, Y, \in g$$

Note : Such an equatioin is called Maurer-Carter Equation.

Proof : From theorem 1 of § 1.12, we know that

$$d\omega(X,Y) = \frac{1}{2}\{X(\omega(Y)) - Y(\omega(Y)) - \omega([X,Y])\} \text{ for every vector field X, Y}$$

If X, Y are in g then by Exercise 2, $\omega(X), \omega(Y)$ are constant functions on G. Hence by Exercise 2 of § 1.4),

$$X.\omega(Y) = 0, \quad Y.\omega(X) = 0$$

Thus the above equation reduces to

$$d\omega(X,Y) = \frac{1}{2}\omega([X,Y]).$$

Exercise : 6. Show that

$$d\omega^i = -\frac{1}{2}\sum_{j,k} c^i_{jk}\ \omega^j \wedge \omega^k = \sum_{j,k} c^i_{jk}\ \omega^k \wedge \omega^j$$

Solution : If $\{X_1, X_2, \ldots, X_n\}$ is a basis of g and $\{\omega^1, \ldots, \omega^n\}$ is the dual basis of g^*, then

1) $\omega^i(X_j) = \delta^i_j$

Hence from theorem 1 above

$$d\omega^i(X_j, X_k) = -\frac{1}{2}\omega^i([X_j, X_k])$$

$$= -\frac{1}{2}\omega^i\left\{\sum c^m_{jk}\ X_m\right\} \text{ from Exercise 2 of Q 2.2}$$

$$= -\frac{1}{2}\Sigma c^m_{jk}\ \omega^i(X_m) = -\frac{1}{2}\Sigma c^m_{jk}\ \partial^i_m$$

$$= -\frac{1}{2}c^i_{jk} \quad \text{by i)}$$

Again from § 1.11

$$\sum_{m,n} c^i_{mn}\,(\omega^m \wedge \omega^n)(X_j, X_k) = \frac{1}{2}\sum_{m,n} c^i_{mn}\,\left\{\omega^m(X_j)\omega^n(X_k) - \omega^m(X_k)\omega^n(X_j)\right\}$$

$$= \frac{1}{2}\sum_{m,n} c^i_{mn}\,\left\{\delta^m_j\,\delta^n_k - \delta^m_k\,\delta^n_j\right\}$$

$$= \frac{1}{2}\left\{c^i_{jk} - c^i_{kj}\right\}$$

$$= \frac{1}{2}\left\{c^i_{jk} + c^i_{jk}\right\} \text{ by i) of Exercise 1 of § 2.2}$$

$$= \frac{1}{2}\cdot 2\,c^i_{jk}$$

$$= c^i_{jk}$$

Thus $d\omega^i(X_j, X_k) = -\dfrac{1}{2}\sum_{m,n} c^i_{mn}\,\omega^m \wedge \omega^n(X_j, X_k), \quad \forall x_j, x_k$

or $\quad d\omega = -\dfrac{1}{2}\sum_{m,n} c^i_{mn}\;\omega^m \wedge \omega^n$

or $\quad d\omega^i = -\dfrac{1}{2}\sum_{j,k} c^i_{j\,k}\;\omega^j \wedge \omega^k$

Take $i, j, k = 1, 2, 3$, then

$$\sum_{j,k} c^i_{j\,k}\;\omega^j \wedge \omega^k = c^i_{12}\,\omega^1 \wedge \omega^2 + c^i_{13}\,\omega^1 \wedge \omega^3 + c^i_{21}\,\omega^2 \wedge \omega^1 + c^i_{23}\,\omega^3 \wedge \omega^2$$

$$+ c^i_{31}\,\omega^3 \wedge \omega^1 + c^i_{32}\,\omega^3 \wedge \omega^2$$

$$= 2\,c^i_{12}\,\omega^1 \wedge \omega^2 + 2\,c^i_{13}\,\omega^1 \wedge \omega^3 + 2\,c^i_{23}\,\omega^2 \wedge \omega^3$$

as $c^i_{j\,k} = -c^i_{h\,k}$

$$= 2\sum_{j<k} c^i_{j\,k}\;\omega^j \wedge \omega^k$$

Thus, we write

$$d\omega^i = -\sum_{j<k} c^i_{j\,k} \;\; \omega^j \wedge \omega^k$$

Hence

$$d\omega^i = +\sum_{j<k} c^i_{j\,k} \;\; \omega^k \wedge \omega^j .$$

§. 2.4 Automorphism :

A mapping, denoted by σ_a for every $a \in \mathrm{G}$, $\sigma_a : \mathrm{G} \to \mathrm{G}$ defined by

$$\sigma_a(x) = axa^{-1}, \quad \forall\ x \in \mathrm{G}$$

is said to be an **inner automorphism** if

i) $\sigma_a(xy) = \sigma_a(x)\,\sigma_a(y)$

ii) σ_a is injective

iii) σ_a is surjective

such σ_a is written as *ada*.

Exercise : Show that if G is a Lie group, $h \in \mathrm{G}$, then the map

$$\mathrm{I}_h : \mathrm{G} \to \mathrm{G}$$

defined by

$$\mathrm{I}_h\,(k) = hkh^{-1}$$

is an automorphism.

An inner automorphism of a Lie group G is defined by

2.16) $(ada)(x) = axa^{-1}$, $\forall x \in \mathrm{G}$

Now, $(L_a R_{a^{-1}})x = L_a(R_{a^{-1}}x) = L_a(xa^{-1}) = axa^{-1} = (ada)(x)$

$\therefore \quad L_a R_{a^{-1}} = ada$

Using 2.3) we get

2.17) $ada = L_a R_{a^{-1}} = R_{a^{-1}} L_a$

Note that *ada* is a diffeomorphism.

Theorem 1 : Every inner automorphism of a Lie group G induces an automorphism of the Lie algebra *g* of G.

Proof : For every $a \in \mathrm{G}$ let us denote the inner automorphism on G by

i) $(ada)(x) = axa^{-1}$, $\forall\ x \in \mathrm{G}$

Now for every G, $e \in \mathrm{G}$ and from § 1.7 such ada : $\mathrm{G} \to \mathrm{G}$ induces a differential mapping $(ada)_*$,

$$(ada)_* : \mathrm{T_e(G)} \to \mathrm{T}_{ada\,(\mathrm{e})}^{(\mathrm{G})} \equiv \mathrm{T_e(G)}$$

Such a mapping is a linear mapping and by Theorem 2 of § 2.2, the Lie subalgebra *g* of *a* Lie group G is such that

$$\mathrm{g} \cong \mathrm{T_e(G)}$$

Thus to show every *ada* induces an automorphism of the Lie algebra *g* of G we are to show

ii) $(ada)_*$ is a mapping from *g* to *g*

iii) $(ada)_*$ is a homomorphism i.e.

$$(ada)_*(X+Y) = (ada)_* X + (ada)_* Y$$

$$(ada)_*(bX) = b(ada)_* X$$

$$(ada)_*[X, Y] = \left[(ada)_* X + (ada)_* Y\right] \quad , \quad \forall\ X, Y \text{ in } g$$

iv) $(ada)_*$ is injective

v) $(ada)_*$ is surjective

ii) Let $Y \in \mathrm{G}$. Then on using 2.17) we get

$$(ada)_* Y = \left(R_{a^{-1}} \circ L_a\right)_* Y \ = \left(R_{a^{-1}}\right)_* \left(L_a\right)_* Y \qquad \text{as} \quad (f \circ g)_* = f_* \circ g_*$$

$$= \left(R_{a^{-1}}\right)_* Y$$

Thus

vi) $(ada)_* = \left(R_{a^{-1}}\right)_*$

Again, $(L_p)_* \left\{\left(R_{a^{-1}}\right)_* Y\right\} = \left\{(L_p)_* \left(R_{a^{-1}}\right)_* Y\right\}$, for every $p \in \mathrm{G}$

$= \left(L_p \circ R_{a^{-1}}\right)_* Y$

$= \left(R_{a^{-1}} \circ L_p\right)_* Y$ by 2.3)

$= \left\{\left(R_{a^{-1}}\right)_* \circ \left(L_p\right)_*\right\} Y$

$= \left(R_{a^{-1}}\right)_* \left(L_p\right)_* Y$

$= \left(R_{a^{-1}}\right)_* Y$ as $Y \in g$

Consequently, from above, it follows that $\left(R_{a^{-1}}\right)_* Y \in g$.

Hence $(ada)_*$ is a mapping from g to g.

iii) From § 1.7) we know that such $(ada)_*$ is a linear mapping

i.e.

$$(ada)_*(X + Y) = (ada)_* X + (ada)_* Y$$

$$(ada)_*(bX) = b(ada)_* X, \quad b \in R$$

Further, such $(ada)_*$ satisfies

$$(ada)_*[X, Y] = \left[(ada)_* X,\ (ada)_* Y\right]$$

Thus $(ada)_*$ is a homomorphism from g to g.

iv) Clearly $(ada)_*$ is injective, on using vi) and the fact that $R_{a^{-1}}$ is a translation on G.

v) For every $a \in \mathrm{G}$, $a^{-1} \in \mathrm{G}$ and we set

$(ada^{-1})_* X = Y$, where $X \in \mathrm{G}$

we will show that $Y \in \mathrm{G}$ and $(ada)_* Y = X$. Now, for $s \in \mathrm{G}$,

$(L_s)_* Y = (L_s)_*(ada^{-1})_* X = (L_s)_*(R_a \circ La^{-1})_* X$ by (2.17)

$= (L_s)_* \left\{(R_a)_* (L_{a^{-1}})_*\right\} X$

$= (L_s)_* \circ (R_a)_* X$

$= (L_s \circ R_a)_* X = (R_a \circ L_s)_* X = (R_a)_* X$

$= (ada^{-1})_* X$

$= Y$ as defined.

Thus $Y \in g$

Finally

$(ada)_* Y = \left(L_a \circ R_{a^{-1}}\right)_* Y$ by (2.17)

$= \left(L_a \circ R_{a^{-1}}\right)_* (ada^{-1})_* X$ as defined

$= \left(L_a \circ R_{a^{-1}}\right)_* \left(R_a \circ L_{a^{-1}}\right)_* X$ by (2.17)

$= \left(L_a \circ R_{a^{-1}} \circ R_a L_{a^{-1}}\right)_* X$ by (1.7)

$= (L_e)_* X$ by (2.3) , where $(L_e)_*$ is the identity differential

$=$ X

Consequently, $(ada)_*$ is a surjective mapping.

Combining ii) — v), we thus claim

$(ada)_* : g \to g$

is a Lie algebra automorphism.

This completes the proof.

Note : We also write

$(ada)_* = \mathrm{A}da$, for every $a \in g$.

and $a \to \mathrm{A}da$

is called the **Adjoint representation** of G to g.

§. 2.5 One parameter subgroup of a Lie group

Let a mapping

$a : \mathrm{R} \to \mathrm{G}$

denoted by $a : t \to a(t)$

be a differentiable curve on G. If for all *s, t* in R

$$a(t)\, a(s) = a(t+s)$$

then the family $\{\, a(t) \mid t \in R \,\}$ is called a one-parameter subgroup of G.

Exercises : 1. Let H $= \{\, a(t) \mid t \in R \,\}$ be a one-parameter subgroup of a Lie group G. Show that H is a commutative subgroup of G.

2. If X is a left invariant vector field on G, prove that, it is complete

We set

2.18) $a(t) = a_t = \phi_t(e)$

where $\{\, \phi_t : t \in \mathrm{R} \,\}$ is one parameter group of transformations on G, generated by the left invariant vector field x.

Exercises : 3. Let $\{\, \phi_t \mid t \in \mathrm{R} \,\}$ be a one-parameter group of transformations on G, generated by $X \in g$ and $\phi_t(e) = a(t)$. If for every $s \in g$,

$$\phi_t \circ L_s = L_s \circ \phi_t$$

show that the set $\{\, a(t) \mid t \in R \,\}$ is a one-parameter subgroup of G and

$\phi_t = R_{a_t}$ holds, for all $t \in R$

4. Let the vector field X be generated by the one parameter group of transformations $\left\{ R_{a_t} \mid t \in R \right\}$ on G. Show that X is left invariant on G.

Solution : As $\{\, \phi_t \mid t \in \mathrm{R} \,\}$ is a one-parameter group of transformations on G and $a : t \in R \to a(t) \in \mathrm{G}$ is a differentiable mapping, by definition

$a(t) \cdot a(s) = L_{a(t)}\big(a(s)\big)$

$= L_{a(t)}\big(\phi_s(e)\big)$, as defined in the hypothesis

$= \big(L_{a(t)} \circ \phi_s\big)(e)$

$= \big(\phi_s \circ L_{a(t)}\big)(e)$ by the hypothesis

$= \phi_s\left(L_{a(t)}{}^{(e)}\right)$

$= \phi_s\left(a(t)e\right)$

$= \phi_s\left(a(t)\right)$

$= \phi_s\left(q_t(e)\right)$ as defined

$= \left(\phi_{s\circ}\phi_t\right)(e)$

$= \phi_{s+t}(e)$ is $\{\phi\,(t)\}$ a one-parameter group of transformations on G

$= \phi_{t+s}(e)$, as $s+t=t+s$ in R

$= a(t+s)$

Thus the set $\{a(t)\,|\,t\in R\}$ is a one-parameter subgroup of G.

Again $\phi_t(s)=\phi_t(se)=\phi_t\left(L_s(e)\right)=(\phi_t\circ L_s)(e)\ = L_s\left(\phi_t(e)\right)=L_s(a_t)$ by (2.18)

$= sa_t$

or $\quad \phi_t(s)=R_{a_t}(s), \qquad \forall s\in \mathrm{G}$

$\therefore\ \phi_t = R_{a_t}$

4. From Exercise 3 above

$R_{a_t} = \phi_t$

As it is given that $\left\{R_{a_t}\ |\, t\in R\ \right\}$ generates the vector field X, from § 1.9, we can say that X_s is the tangent vector to the curve R_{a_t} and we write

$$X_s f = \lim_{t\to 0}\frac{1}{t}\left\{f\left(R_{a_t}(s)\right)-f(s)\right\}$$

$$= \lim_{t\to 0}\frac{1}{t}\left\{f\left(L_q(R_{a_t}(q^{-1}s))\right)-f(L_q)(q^{-1}s)\right\}$$

$$= \lim_{t \to 0} \frac{1}{t} \left\{ f\left(L_q(R_{a_t}(q^{-1}s))\right) - (f \circ L_q)(q^{-1}s) \right\}$$

$$= \lim_{t \to 0} \frac{1}{t} \left\{ (f \circ L_q)\left(R_{a_t}(q^{-1}s)\right) - (f \circ L_q)(q^{-1}s) \right\}$$

i) $X_s f = X_{q^{-1}s}(f \circ L_q)$ from § 1.9

We are left to prove that $X \in g$. Note that, for $q \in g$.

$$L_q : G \to G$$

is a left translation on G and $(L_q)_* : T_p(G) \to T_{L_q(p)}(G) \equiv T_{qp}(G)$ is its differential. Hence

$$\left((L_q)_* X\right) f = X_p(f \circ L_q) \text{ by § 1.7, where } f \in F(G)$$

or $$\left((L_q)_* X\right)_{L_q(p)} f = X_p(f \circ L_q)$$

If $L_q(p) = s$, then $p = L_q^{-1}(s) = L_{q^{-1}}(s)$ by (2.5)

$$\therefore \quad p = q^{-1}s$$

Consequently, the above equation reduces to

$$\left((L_q)_* X\right)_s f = X_{q^{-1}s}(f \circ L_q) = X_s f \qquad \text{by i)}$$

$$\therefore \quad \left((L_q)_* X\right)_s = X_s, \quad \forall\, s \in G$$

$\therefore \quad (L_q)_* = X$, which shows that X is left invariant.

Theorem 1 : If $X, Y \in g$, then

$$[Y, X] = \lim_{t \to \circ} \frac{1}{t} \left\{ \left(Ada_t^{-1}\right)Y - Y \right\}$$

Proof : Every $X \in g$ induces $\{\phi_t | t \in R\}$ as its 1-parameter group of transformations on G. Hence by §1.9.

$$[Y, X] = -[X, Y] = \lim_{t \to \circ} \frac{1}{t} \left\{ (\phi_t)_* Y - Y \right\}$$

Now from § 2.4

$$\left(\mathrm{A}da_t^{-1}\right)\mathrm{Y} = \left(ada_t^{-1}\right)_*\mathrm{Y}$$

$$= \left(\mathrm{R}_{a_t} \circ \mathrm{L}_{a_t^{-1}}\right)_*\mathrm{Y} \text{ by } 2.17)$$

$$= \left(\mathrm{R}_{a_t}\right)_*\left\{\left(\mathrm{L}_{a_t^{-1}}\right)_*\mathrm{Y}\right\}$$

$$= \left(\mathrm{R}_{a_t}\right)_*\mathrm{Y} \quad \text{, as } \mathrm{Y} \in \mathrm{g}$$

$$= (\phi_t)_*\mathrm{Y} \text{ by Exercise 3.}$$

Consequently, the above question reduces to,

$$[\mathrm{Y}, \mathrm{X}] = \lim_{t \to \circ} \frac{1}{t}\left\{\left(\mathrm{A}da_t^{-1}\right)\mathrm{Y} - \mathrm{Y}\right\}$$

§ 2.6 Lie Transformation group (Action of a Lie group on a Manifold)

A Lie group G is a Lie transformation group on a manifold M or G is said to act differentiably on M if the following conditions are satisfied :

i) Each $a \in \mathrm{G}$ induces a transformation on M, denoted by

$$p \to pa, \quad \forall p \in \mathrm{M}.$$

ii) (a, p) : $\mathrm{G} \times \mathrm{M} \to \mathrm{p}a \in \mathrm{M}$ is a differentiable map.

iii) $p(ab) = (pa)b \quad , \quad \forall\, a, b \in \mathrm{G},\ p \in \mathrm{M}.$

We say that **G acts on M on the right.**

Similarly, the action of G on the left can be defined.

Exercise : 1. Let $\mathrm{G} = \mathrm{GL}_2(\mathrm{R})$ and M = R and

$$\theta : \mathrm{G} \times \mathrm{M} \to \mathrm{M}$$

be a differentiable mapping defined by

$$\theta\left(\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}, p\right) = ap + b, \quad a > 0, \quad a, b \in \mathrm{R}$$

Show that θ is an action on M.

Solution : In this case, $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{G}$ and

i) $\theta\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, p\right) = 1 \cdot p + 0, \; = p$

ii) $\left(\theta\begin{pmatrix} a' & b' \\ 0 & 1 \end{pmatrix}, \left(\theta\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}, p\right)\right) = \left(\theta\begin{pmatrix} a' & b' \\ 0 & 1 \end{pmatrix}, ap+b\right)$ as defined

$= a'(ap+b) + b',$ as defined

$= a'ap + a'b + b',$

$= \theta\left(\begin{pmatrix} aa' & a'b+b' \\ 0 & 1 \end{pmatrix}, p\right)$ as defined

$= \theta\left(\begin{pmatrix} a' & b' \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}, p\right)$

Thus θ is an action on M.

Definition : If G acts on M on the right such that

2.19) $pa = p, \; \forall p \in \mathrm{M}$ implies that $a = e$

then, G is said to act **effectively** on M.

Note : There is no transformation, other than the identity one, which leaves every point fixed.

If G acts on M on the right such that

2.20) $pa = p, \; \forall p \in \mathrm{M}$, implies that $a = e$ for some $p \in \mathrm{M}$ then, G is said to act **freely** on M.

Note : In this case, it has isolated fixed points.

Theorem 1 : If G acts on M, then the mapping

$$\sigma : g \to \chi(\mathrm{M})$$

denoted by

$$\sigma : \mathrm{A} \to \sigma(\mathrm{A}) = \mathrm{A}^*$$

is a Lie Algebra homomorphism

Note : $\sigma(\mathrm{A})$ is called the **fundamental vector field** on M corresponding to $\mathrm{A} \in g$.

Proof : For every $\mathrm{p} \in \mathrm{G}$ let

$$\sigma_p : \mathrm{G} \to \mathrm{M}$$

be a mapping such that

i) $\sigma_p(a) = pa$

Such a mapping is called the **fundamental map** corresponding to $\mathrm{p} \in \mathrm{M}$.

We want to show that

$$\sigma : g \to \chi(\mathrm{M})$$

is a Lie Algebra homomorphism i.e. we are to prove

ii) $\sigma(\mathrm{X}+\mathrm{Y}) = \sigma(\mathrm{X}) + \sigma(\mathrm{Y})$

iii) $\sigma(\mathrm{bX}) = \mathrm{b}\sigma(\mathrm{X}), \mathrm{b} \in \mathrm{R}$

iv) $\sigma[\mathrm{X}, \mathrm{Y}] = [\sigma\mathrm{X}, \sigma\mathrm{Y}]$

It is evident from i) that

v) $\sigma_p(a) = pa = \mathrm{R}_a(p)$

Let $\mathrm{A} \in \mathrm{g}$. Then from §2.5, A generates $\{\phi_t \mid t \in \mathrm{R}\}$ as its 1-parameter group of transformation on G, such that

$$a(t) \equiv a_t = \phi_t(e)$$

In this case, such $a(t)$ is the integral curve of A on G. The map

$$(\sigma_p)_* : T_e(G) \to T_{\sigma_p(e)}(M) \equiv T_p(M)$$

is the differential map of σ_p and is a linear mapping by definition such that $(\sigma_p)_* \mathrm{X}_e \in \mathrm{T}_p(\mathrm{M})$.

Using the hypothesis of the theorem

vi) $\left(\sigma_p\right)_* \mathrm{A}_e = \{\sigma(\mathrm{A})\}_{\sigma_p(e)} = \{\sigma(\mathrm{A})\}_p = \mathrm{A}_p^*$

Note that for every A, B, in g, A + B is in g and hence

$$\{\sigma(A+B)\}_p = (\sigma_p)_*(A+B)_e = (\sigma_p)_*(A_e + B_e) = (\sigma_p)_* A_e + (\sigma_p)_* B_e, \text{ as } (\sigma_p)_* \text{ is linear}$$

$$= \{\sigma(A)\}_p + \{\sigma(B)\}_p$$

$\therefore \quad \sigma(A+B) = \sigma(A) + \sigma(B), \quad \forall\ p \in M.$

Also for $b \in R$ $bA \in g$ and hence

$$\{\sigma(bA)\}_p = (\sigma_p)_*(bA)_e = (\sigma_p)_*(A)_e = b(\sigma_p)_* A_e = b\{\sigma(A)\}_p$$

$\therefore \quad \sigma(bA) = b\sigma(A)$

Thus σ is a linear mapping

Now A_e is the tangent vector to the curve $a(t) \equiv a_t$ at $a(0) = e$. Consequently by §1.7, the vector field $(\sigma_p)_* A_e \in T_{\sigma_p(e)}(M) \equiv T_p(M)$ is defined to be the tangent vector to the curve $\sigma_p(a_t) = pa_t = R_{a_t}(p)$ at $\sigma_p(a_o) = \sigma_p(e) = p$. consequently, by vi), we see that A_e^* induce $R_{a_t} p$ as its one-parameter group of transformations on M.

Again $[\sigma(A), \sigma(B)]_p = [A^*, B^*]_p$

$$= \lim_{t \to 0} \frac{1}{t}\left\{ B_p^* - \left((R_{a_t})_* B^*\right)_p \right\} \text{ by Theorem 3 of §1.9}$$

$$= \lim_{t \to 0} \frac{1}{t}\left\{ (\sigma_p)_* B_e - (R_{a_t})_* B_q^* \right\} \text{ say, where}$$

vii) $\quad p = R_{a_t}(q)$

viii) $\quad$ or $q = (R_{a_t})^{-1} p = R_{a_t^{-1}}(p) = pa_t^{-1}$

Thus $(R_{a_t})_* B_q^* = (R_{a_t})_* B^*_{pa_t^{-1}}$ by vii) above

$$= (R_{a_t})_* (\sigma_{pa_t^{-1}})_* B_e \text{ by vi)}$$

$$= (R_{a_t} \circ \sigma_{pa_t^{-1}})_* B_e \text{ where } R_{a_t} \circ \sigma_{pa_t^{-1}} : G \to M$$

Hence for $b \in G$

$$\left(R_{a_t} \circ \sigma_{pa_t^{-1}}\right)(b) = R_{a_t}\left(\sigma_{pa_t^{-1}}(b)\right)$$

$$= R_{a_t}\ (pa_t^{-1}b) \text{ by i)}$$

$$= pa_t^{-1}ba_t \text{ by definition}$$

$$= \sigma_p\left(a_t^{-1}\, ba_t\right) \text{ by i)}$$

$$= \sigma_p\left(ada_t^{-1}(b)\right) \text{ by 2.16) of § 2.4}$$

$$= \left(\sigma_p \circ ada_t^{-1}\right)(\mathrm{b})$$

$$\therefore \qquad R_{a_t} \circ \sigma_{pa_t^{-1}} = \sigma_p \circ ada_t^{-1}$$

Consequently, $\left(R_{a_t}\right)_* B_q^* = \left(R_{a_t} \circ \sigma_{pa_t^{-1}}\right)_* B_e$ reduces to

$\left(R_{a_t}\right)_* B_q^* = \left(\sigma_p \circ ada_t^{-1}\right)_* B_e = \left(\sigma_p\right)_*\left(\left(ada_t^{-1}\right)_* B_e\right) = \left(\sigma_p\right)_*\left(\left(\mathrm{A}da_t^{-1}\right)_* B_e\right)$ from the Note of § 2.4

Thus we find

$$\left[\sigma(\mathrm{A}), \sigma(\mathrm{B})\right]_p = \lim_{t \to 0} \frac{1}{t}\left\{(\sigma_p)_* B_e - \left(\sigma_p\right)_*\left(\left(\mathrm{A}da_t^{-1}\right)_* B_e\right)\right\}$$

$$= \left(\sigma_p\right)_* \lim_{t \to 0} \frac{1}{t}\left\{B_e - \left(\mathrm{A}da_t^{-1}\right)_* B_e\right\} \text{ as } \left(\sigma_p\right)_* \text{ is a linear mapping.}$$

$$= \left(\sigma_p\right)_* [\mathrm{A}, \mathrm{B}]_e \text{ by § 1.9}$$

$$= \left(\sigma[\mathrm{A}, \mathrm{B}]\right)_p \text{ by vi)}$$

$$\therefore \qquad \sigma[\mathrm{A}, \mathrm{B}] = \left[\sigma(\mathrm{A}), \sigma(\mathrm{B})\right]$$

Thus the mapping

$$\sigma : g \to \chi(\mathrm{M})$$

is a Lie Algebra homomorphism.

Theorem 2 : If G acts effectively on M, then the map

$$\sigma : g \to \chi(\mathrm{M})$$

defined by

$$\sigma : \mathrm{A} \to \sigma(\mathrm{A}) = \mathrm{A}^*$$

is an isomorphism.

Proof : From Theorem 1, we know that such map $\sigma : g \to \chi(\mathrm{M})$ is a Lie Algebra homomorphism. Hence we are left to prove that

i) σ is injective and ii) σ is surjective.

i) Let A, B $\in g$ and $\sigma(\mathrm{A}) = \sigma(\mathrm{B})$ Then

$\sigma(\mathrm{A}-\mathrm{B}) = \theta,$ as σ is a linear mapping.

or $(\mathrm{A}-\mathrm{B})^* = \theta$

i.e. $(\mathrm{A}-\mathrm{B})^*$ is the null vector on M. Now A–B $\in g$ and it will generate $\{\, \psi_t(e) \mid t \in \mathrm{R} \,\}$ as its 1-parameter group of transformations on G such that $(\mathrm{A}-\mathrm{B})_e$ is the tangent vector to the curve, say

$$b(t) = b_t = \psi_t(e) \text{ at } b(o) = e$$

Consequently, the vector field $(\mathrm{A}-\mathrm{B})^* = (\sigma_p)_*(\mathrm{A}-\mathrm{B})_e$ is the tangent vector to the curve

$$\sigma_p(b(t)) = pb_t = R_{b_t}(p) \text{ at } \sigma_p(b(o)) = \sigma_p(e) = pe = p.$$

Thus $(\mathrm{A}-\mathrm{B})^* = (\sigma_p)_*(\mathrm{A}-\mathrm{B})_e$ generates $\{\mathrm{R}_{b_t}(p) \mid t \in \mathrm{R}\}$ as its 1-parameter group of transformations on M. But $(\mathrm{A}-\mathrm{B})^*$ is the null vector on M. Hence the integral curve of $(\mathrm{A}-\mathrm{B})^*$ will reduce to a single point of itself. Thus

$$\mathrm{R}_{b_t}(p) = p$$

or $pb_t = p$

As G acts effectively on M, comparing this with 2.19) we get, $b_t = e, \ \forall\, p \in \mathrm{M}$.

Again $(L_q)_*(\mathrm{A}-\mathrm{B}) = \mathrm{A}-\mathrm{B}$ as $(\mathrm{A}-\mathrm{B}) \in g$

$\therefore\ L_q \circ \psi_t = \psi_t \circ L_q$ from § 1.9

Thus $\psi_t(q) = \psi_t(qe) = \psi_t\big(L_q(e)\big) = (\psi \circ L_q)(e) = (L_q \circ \psi_t)(e) = L_q(b_t)$

$$= qb_t = q_e = q$$

Hence from § 1.9

$$(A-B)_q f = \lim_{t \to 0} \frac{1}{t}\left\{ f\big(\psi_t(q)\big) - f(q) \right\} \text{ reduces to}$$

$$(A-B)_q f = \lim_{t \to 0} \frac{1}{t}\left\{ f(q) - f(q) \right\} = \theta.$$

Thus $\quad A - B = \theta$

i.e. A = B.

Hence $\sigma(A) = \sigma(A)$ implies that A = B. Consequently σ is injective.

ii) As G acts effectively on M, σ is surjective.

Thus the map is a Lie Algebra isomorphism and this completes the proof.

Theorem 3 : If G acts freely on M, then, for every non-zero vector field $A \in g$, the vector field A^* on M can never vanish.

Proof : If possible, let A^* be a null vector on M. Then, as done in the previous theorem, every $A \in$ g will generate $\{\psi_t(e) \mid t \in R\}$ as its 1-parameter group of transformations on G and we will have

$$\psi_t(q) = q$$

Consequently from the definition, as given in § 1.9

$$A_q f = \left[\frac{d}{dt} f\big(\phi_t(q)\big)\right]_{t=0}$$

$$= \lim_{t \to 0} \frac{f\big(\psi_t(q)\big) - f(q)}{t}$$

$$= 0.$$

Hence A becomes a null vector, contradicting the hypothesis. Thus the vector field A^* on M can never vanish.

REFERENCE

1. P. M. Cohn : Lie groups
2. B. B. Sinha : An Introduction to Modern Different geometry
3. S. Helgason : Differential geometry, Lie groups and Symmetric spaces.

UNIT - 3

3.1 Linear Connection :

The concept of linear (affine) connection was first defined by Levi-Civita for Riemannian manifolds, generalising the notion of parallelism for Eucliden Spaces. This definition is given in the sense of KOSZUL.

A linear connection on a manifold M is a mapping

$$\nabla : \chi(M) \times \chi(M) \to \chi(M)$$

denoted by

$$\nabla : (X, Y) \to \nabla_X Y$$

satisfying the following conditions :

i) $\nabla_X (Y + Z) = \nabla_X Y + \nabla_X Z$

ii) $\nabla_{(Y+Z)} X = \nabla_Y X + \nabla_Z X$

iii) $\nabla_{fX} Y = f \nabla_X Y$

iv) $\nabla_X (f Y) = (Xf)Y + f \nabla_X Y, \quad \forall X, Y, Z \in \chi(M), \; f \in F(M)$

The vector field $\nabla_X Y$ is called the covariant derivative of Y in the direction of X with respect to the connection

If P is a tensor field of type (o, s) we define

v) $\nabla_X P = XP, \quad \text{if } s = o$

vi) $(\nabla_X P)(Y_1, Y_2, \ldots, Y_n) = X\left(P\left(Y_1, Y_2, \ldots, Y_n\right)\right) - \sum_{i=1}^{s} P\left(Y_1, \ldots, \nabla_X Y_i, \ldots, Y_s\right)$

Exercise 1 : Let M = R^n and X, Y, $\in \chi(M)$ be such that

$$Y = \sum_{i=1}^{n} b^i \frac{\partial}{\partial x^i} \quad \text{where} \quad \nabla_X Y = \left(Xb^i\right)\frac{\partial}{\partial x^i}$$

Show that ∇ determines a linear connection on M.

Solution : Let $X = a^i \frac{\partial}{\partial x^i}$, $Z = c^i \frac{\partial}{\partial x^i}$ with a^i , $c^i \in F(M)$, $i = 1, \ldots, n$

Then i) $\nabla_X (Y + Z) = \left(X\left(b^i + c^i\right)\right) \frac{\partial}{\partial x^i}$, as defined

$$= \left(Xb^i + Xc^i\right) \frac{\partial}{\partial x^i} = \left(Xb^i\right) \frac{\partial}{\partial x^i} + \left(Xc^i\right) \frac{\partial}{\partial x^i}$$

$$= \nabla_X Y + \nabla_X Z$$

Similarly it can be shown that

$$\nabla_{(Y+Z)} X = \nabla_Y X + \nabla_Z X$$

Again, $\nabla_{fX}^{Y} = \left((f\,X)b^i\right) \frac{\partial}{\partial x^i} = \left(f\left(Xb^i\right)\right) \frac{\partial}{\partial x^i}$ as $(f\,Y)h = f\,(Yh)$

$= f\,\nabla_X Y$ and

$\nabla_X (f\,Y) = \left(X\left(f\,b^i\right)\right) \frac{\partial}{\partial x^i}$ as $= \left((X f)b^i + f\,(Xb^i)\right) \frac{\partial}{\partial x^i}$ as $X(fg) = (Xf)g + f\,(Xg)$

$$= (X f)b^i \frac{\partial}{\partial x^i} + f\,(Xb^i) \frac{\partial}{\partial x^i}$$

$$= (X f)\,Y + f\,\nabla_X Y$$

Thus ∇ determines a linear connection on M.

Let $(x^1, x^2, \ldots, x^n)$ be a system of co-ordinates in a neighbourhood U of p of M. We define

3.1) $\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \Gamma_{ij}^{k} \frac{\partial}{\partial x^k}$ where $\Gamma_{ij}^{k} \in F(M)$

Such Γ_{ij}^{k} are called the christoffel symbols or the connection co-efficients or the components of the connection.

Hence if

$$X = \xi^i \frac{\partial}{\partial x^i} \;,\quad Y = \eta^j \frac{\partial}{\partial x^j} \text{ where each } \xi^i, \eta^j \in F(M),\ i = 1, \dots, n \text{ we see that}$$

$$\nabla_X Y = \nabla_{\xi^i \frac{\partial}{\partial x^i}} \left(\eta^j \frac{\partial}{\partial x^j} \right)$$

$$= \xi^i \nabla_{\frac{\partial}{\partial x^j}} \left(\eta^j \frac{\partial}{\partial x^j} \right) \text{ by iii)}$$

$$= \xi^i \left(\frac{\partial \eta^j}{\partial x^i} \cdot \frac{\partial}{\partial x^j} + \eta^j \Gamma_{ij}^k \frac{\partial}{\partial x^k} \right) \text{ by iv) and 3.1)}$$

3.2) $$\nabla_X Y = \left(\xi^i \frac{\partial \eta^k}{\partial x^i} + \xi^i \eta^j \Gamma_{ij}^k \right) \frac{\partial}{\partial x^k}$$

Exercise 2 : Let Γ_{ij}^k and $\bar{\Gamma}_{ij}^k$ be the connection co-efficients of the linear connection ∇ with respect to the local coordinate system $(x^1, \dots, x^n)$ and $(y^1, \dots, y^n)$ respectively. Show that in the intersection of the two coordinate neighbourhoods

$$\bar{\Gamma}_{ij}^k = \frac{\partial^2 x^l}{\partial y^i \partial y^j} \cdot \frac{\partial y^k}{\partial x^l} + \Gamma_{rs}^t \frac{\partial x^r}{\partial y^i} \cdot \frac{\partial x^r}{\partial y^j} \cdot \frac{\partial y^k}{\partial x^t}$$

Solution : In the intersection of the two coordinates

$$\frac{\partial}{\partial y^j} = \frac{\partial x^l}{\partial y^j} \cdot \frac{\partial}{\partial x^l}$$

or $$\frac{\partial y^j}{\partial x^s} \cdot \frac{\partial}{\partial y^j} = \frac{\partial y^j}{\partial x^s} \cdot \frac{\partial x^l}{\partial y^j} \cdot \frac{\partial}{\partial x^l} = \frac{\partial}{\partial x^s}$$

Again, from 3.1) we see that

$$\bar{\Gamma}_{ij}^k \frac{\partial}{\partial y^k} = \nabla_{\frac{\partial}{\partial y^i}} \frac{\partial}{\partial y^j} = \nabla_{\frac{\partial}{\partial y^i}} \left(\frac{\partial x^l}{\partial y^j} \cdot \frac{\partial}{\partial x^l} \right) \text{ from above}$$

$$= \frac{\partial^2 x^l}{\partial y^i \partial y^j} \cdot \frac{\partial}{\partial x^l} + \frac{\partial x^l}{\partial y^j} \nabla_{\frac{\partial}{\partial y^i}} \frac{\partial}{\partial x^l} \qquad \text{by iv)}$$

$$= \frac{\partial^2 x^l}{\partial y^i \partial y^j} \cdot \frac{\partial}{\partial x^l} + \frac{\partial x^l}{\partial y^j} \nabla_{\frac{\partial x^s}{\partial y^i} \cdot \frac{\partial}{\partial x^s}} \frac{\partial}{\partial x^l} \qquad \text{from above}$$

$$= \frac{\partial^2 x^l}{\partial y^i \partial y^i} \cdot \frac{\partial}{\partial x^l} + \frac{\partial x^l}{\partial y^j} \cdot \frac{\partial x^s}{\partial y^i} \nabla_{\frac{\partial}{\partial x^s}} \frac{\partial}{\partial x^l} \quad \text{by iii)}$$

$$= \frac{\partial^2 x^l}{\partial y^i \partial y^j} \cdot \frac{\partial}{\partial x^l} + \frac{\partial x^l}{\partial y^j} \cdot \frac{\partial x^s}{\partial y^i} \Gamma_{sl}^k \frac{\partial}{\partial x^k} \quad \text{by 3.1)}$$

$$= \frac{\partial^2 x^l}{\partial y^i \partial y^j} \cdot \frac{\partial y^k}{\partial x^l} \cdot \frac{\partial}{\partial y^k} + \frac{\partial x^r}{\partial y^i} \cdot \frac{\partial x^s}{\partial y^i} \Gamma_{rs}^t \frac{\partial}{\partial x^t}$$

Changing s $\to$ r
$l \to$ s
$k \to$ t

$$= \frac{\partial^2 x^l}{\partial y^i \partial y^j} \cdot \frac{\partial y^k}{\partial x^l} \cdot \frac{\partial}{\partial y^k} + \Gamma_{rs}^t \frac{\partial x^r}{\partial y^i} \cdot \frac{\partial x^s}{\partial y^j} \cdot \frac{\partial y^k}{\partial x^t} \cdot \frac{\partial}{\partial y^k} \quad \text{from above}$$

$$= \left(\frac{\partial^2 x^l}{\partial y^i \partial y^j} \cdot \frac{\partial y^k}{\partial x^l} \cdot + \Gamma_{rs}^t \frac{\partial x^r}{\partial y^i} \cdot \frac{\partial x^s}{\partial y^j} \cdot \frac{\partial y^k}{\partial x^t} \cdot \frac{\partial}{\partial y^k} \right) \frac{\partial}{\partial y^k}$$

Since $\left\{ \frac{\partial}{\partial y^k} : k = 1 \cdots n \right\}$ is a basis of the tangent space and hence linearly independent and the result follows immediately.

3.2 Torsion tensor field and curvature tensor field on a linear connection

we define a mapping

T $: \chi(M) \times \chi(M) \to \chi(M)$ by

3.2) $\mathrm{T}(\mathrm{X}, \mathrm{Y}) = \nabla_{\mathrm{X}}^{\mathrm{Y}} - \nabla_{\mathrm{Y}}^{\mathrm{X}} - [\mathrm{X}, \mathrm{Y}]$

and another

R $: \chi(M) \times \chi(M) \times \chi \to \chi(M)$

3.3) $R(X, Y)Z = \nabla_X \nabla_Y^Z - \nabla_Y \nabla_X^Z - \nabla_{[X, Y]}^Z$

Then T is a tensor field of type (1,2) and is called the torsion tensor field and R is a tensor field of type (1, 3), called the curvature tensor field of M.

A linear connection is said to be symmetric if

3.4) $T(X, Y) = 0$

In such case

3.5) $[X, Y] = \nabla_X^Y - \nabla_Y^X$

Exercise : 1. Verify that

i) $T(X, Y) = -T(Y, X)$;

ii) $T(fX + gY, Z) = fT(X, Z) + gT(Y, Z)$;

iii) $T(fX, gY) = fg\, T(X, Y)$.

2. If $\overline{\nabla}_X^Y = \nabla_X^Y - T(X, Y)$, show that $\overline{\nabla}$ is a linear connection and $\overline{T} = -T$

3. Show that

i) $T(T(X, Y), Z) = T(\nabla_X^Y, Z) + T(Z, \nabla_Y^X) - T([X, Y], Z)$

ii) $R(X, X)Y = 0; R(X, Y)Z = -R(Y, X)Z; \quad R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0$

iii) $R(T(X, Y), Z) = R(\nabla_X^Y, Z) + R(Z, \nabla_Y^X) - R([X, Y], Z)$

iv) $R(X, fY)Z = R(fY, Y)Z = R(X, Y)fZ = f\ R(X, Y)Z$

Hence Show that

$$R(fX, gY)hZ = fgh\ R(X, Y)Z$$

4. **Exercise 3 :** Prove Ricci Identity

a) for a 1-form w :

$$\left(\nabla_X \nabla_Y^\omega - \nabla_Y \nabla_X^\omega - \nabla_{[X, Y]}^\omega\right)Z = -W(R(X, Y)Z)$$

b) for a 2-form W :

$$\left(\nabla_X \nabla_Y^W - \nabla_Y \nabla_X^W - \nabla_{[X,Y]}^W\right)(Z,P) = -W\left(R(X,Y)Z,P\right) - W\left(Z, R(X,Y)P\right)$$

5. If $\left(x^1,\cdots,x^n\right)$ is a local coordinate system and

$$T\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = T_{ij}^k \frac{\partial}{\partial x^k}, R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^j}\right)\frac{\partial}{\partial x^k} = R_{ijk}^h \frac{\partial}{\partial x^h}$$

Show that

i) $T_{ij}^k = \Gamma_{ij}^k - \Gamma_{ji}^k$ and $\Gamma_{ij}^k = \Gamma_{ij}^k$ for a symmetric linear connection

ii) $R_{ijm}^k = \frac{\partial}{\partial x^i}\Gamma_{jm}^k - \frac{\partial}{\partial x^j}\Gamma_{im}^k + \Gamma_{jm}^t \Gamma_{ti}^k - \Gamma_{im}^t \Gamma_{jt}^k$

Solution : 1 i) From the definition

$$T(Y,X) = \nabla_Y X - \nabla_X Y - [Y,X]$$

$$= \nabla_Y X - \nabla_X Y + [X,Y]$$

$$= -\left(\nabla_X Y - \nabla_Y X - [X,Y]\right)$$

$$= -T(X,Y)$$

Thus T is skew-symmetric

ii) $\quad T(fX+gY,Z) = \nabla_{fX+gY}Z - \nabla_Z(fX+gY) - [fX+gY,Z]$

$$= f\,\nabla_X Z + g\nabla_Y Z - \nabla_Z(fX) - \nabla_Z(gY) - [fX,Z] - [gY,Z]$$

$$= g[Y,Z] + (Zg)Y$$

$$= f\left\{\nabla_X Z - \nabla_Z X - [X,Z]\right\} + g\left\{\nabla_Y Z - \nabla_Z Y - [Y,Z]\right\}$$

$$= fT(X,Z) + gT(Y,Z)$$

Again, using the definition, given in § 3.1 and also from § 1.5 we get

Thus T is a bilinear mapping.

2. To prove that $\overline{\nabla}$ is a linear connection, we have to prove i), ii), iii), iv) of § 3.1. Now

$$\overline{\nabla}_X (Y+Z) = \nabla_X (Y+Z) - T(X, Y+Z) \text{ as defined}$$

$$= \nabla_X Y + \nabla_X Z - T(X,Y) - T(X,Z)$$

$$= \overline{\nabla}_X Y + \overline{\nabla}_X Z, \text{ as defined}$$

similarly, other results can be proved and hence $\overline{\nabla}$ is a linear connection. Now,

$$\overline{T}(X,Y) = \overline{\nabla}_X Y + \overline{\nabla}_Y X - [X,Y], \text{ by definition}$$

$$= \nabla_X Y - T(X,Y) - \nabla_Y X + T(Y,X) - [X,Y] \text{ , as defined}$$

$$= T(X,Y) - T(X,Y) - T(X,Y) \text{ by Ex 1 (i) above}$$

$$= -T(X,Y)$$

$$\therefore \overline{T} = -T$$

3. (iv) From the definition

$$R(X, fY)Z = \nabla_X \nabla_{fY} Z - \nabla_{fY} \nabla_X Z - \nabla_{[X,fY]} Z$$

$$= \nabla_X (f\nabla_Y Z) - f\nabla_Y \nabla_X Z - \nabla_{f[XY]+(XfY)} Z$$

$$= (Xf)\nabla_Y Z + f\nabla_X \nabla_Y Z - f\nabla_Y \nabla_X Z - f\nabla_{[X,Y]} Z - (Xf)\nabla_Y Z$$

$$= f\left(\nabla_X \nabla_Y - \nabla_Y Z \nabla_X Z - \nabla_{[X,Y]} Z\right)$$

$$= fR(X, Y)Z \text{ by definition.}$$

5. From the given condition

$$T\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = \nabla_{\frac{\partial}{\partial x^i}}\left(\frac{\partial}{\partial x^j}\right) - \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i} - \left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right]$$

Using 3.1) we find

$$= \Gamma_{ij}^k \frac{\partial}{\partial x^k} - \Gamma_{ji}^k \frac{\partial}{\partial x^k} - 0$$

or, $T_{ji}^k \dfrac{\partial}{\partial x^k} = \left(\Gamma_{ij}^k - \Gamma_{ji}^k\right)\dfrac{\partial}{\partial x^k}$, as defined

Since $\left\{\dfrac{\partial}{\partial x^k} : k = 1, \cdots\cdots, n\right\}$ is a basis and hence linearly independent and thus

i) $\Gamma_{ij}^k = \Gamma_{ij}^k - \Gamma_{ji}^k$

If the linear connection is symmetric, then T = 0. consequently, the above equation reduces to

$$\Gamma_{ij}^k = \Gamma_{ji}^k$$

ii) From the definition, we see that

$$R\left(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^m}=\nabla_{\frac{\partial}{\partial x^i}}\nabla_{\frac{\partial}{\partial x^j}}\frac{\partial}{\partial x^m}-\nabla_{\frac{\partial}{\partial x^j}}\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^m}-\nabla_{\left[\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\right]}\nabla\frac{\partial}{\partial x^m}$$

$$=\nabla_{\frac{\partial}{\partial x^i}}\left(\Gamma^k_{jm}\frac{\partial}{\partial x^k}\right)-\nabla_{\frac{\partial}{\partial x^j}}\left(\Gamma^k_{im}\frac{\partial}{\partial x^k}\right) \text{ as } \left[\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\right]=0$$

$$=\left(\frac{\partial}{\partial x^i}\Gamma^k_{jm}\right)\frac{\partial}{\partial x^k}+\Gamma^k_{jm}\Gamma^t_{ik}\frac{\partial}{\partial x^t}-\left(\frac{\partial}{\partial x^j}\Gamma^k_{im}\right)\frac{\partial}{\partial x^k}+\Gamma^k_{im}\frac{\partial}{\partial x^t}$$

Changing the dummy indices $t\to k, k\to t$ in the 2nd and 4th term we get

$$R^k_{ijm}\frac{\partial}{\partial x^k}=\left(\frac{\partial}{\partial x^i}\Gamma^k_{jm}\right)\frac{\partial}{\partial x^k}+\Gamma^t_{jm}\Gamma^k_{it}\frac{\partial}{\partial x^k}-\frac{\partial}{\partial x^j}\Gamma^k_{jm}\frac{\partial}{\partial x^k}-\Gamma^t_{im}\Gamma^k_{jt}\frac{\partial}{\partial x^k}$$

Since $\left\{\frac{\partial}{\partial x^k}: k=1,\cdots\cdots, n\right\}$ is a basis and hence linearly independent, we get from above

$$R^k_{ijm}=\frac{\partial}{\partial x^i}\Gamma^k_{jm}-\frac{\partial}{\partial x^j}\Gamma^k_{im}+\Gamma^t_{jm}\Gamma^k_{it}-\Gamma^t_{im}\Gamma^k_{tj}$$

3.2 Covariant Differential of A Tensor Field of type (o, s)

The covariant differential of a tensor field of type (0, s) is a tensor field of type (0, s + 1) and is defined as

3.6) $(\nabla P)(X_1, X_2, \cdots\cdots, X_{X+1})=\left(\nabla_{X_{x+1}}P\right)(X_1, X_2, ;\cdots\cdots, X_S)$

Exercise : 1 Let $\Im^i$ be the components of a vector field Y with respect to a local coordinate system $(x^1,\cdots\cdots,x^n)$ i.e. $Y=\Im^i\frac{\partial}{\partial x^i}$

If $\Im^i{}_{,j}$ be the components of the convariant differential ∇Y, so that $\nabla_{\frac{\partial}{\partial x^i}}Y=\Im^i{}_{,j}\frac{\partial}{\partial x^i}$

then, show that

$$\Im^i{}_{,j}=\frac{\partial\Im^i}{\partial x^j}+\Gamma^i_{kj}\Im^k$$

2. Let ω be a 1 form and $\omega_l d_x l$

If we write

$$\nabla_{\frac{\partial}{\partial x^i}}\omega\left(\frac{\partial}{\partial x^k}\right)=\omega_{k,i}$$

show that

$$\omega_{k,i} - \frac{\partial \omega_k}{\partial x^i} - \omega_h \Gamma^h_{ki}$$

3. If we write $\left(\nabla_{\frac{\partial}{\partial x^k}} R\right)\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^m} = R^h_{ijm,k}\frac{\partial}{\partial x^h}$

show that $R^h_{ijm,k} = \frac{\partial}{\partial x^k} R^h_{ijm} + R^s_{ijm}\Gamma^h_{sk} - R^h_{sjm}\Gamma^s_{ik} - R^h_{ism}\Gamma^s_{jk} - R^h_{ijs}\Gamma^s_{mk}$

Solution : 1. We write

$$\nabla_{\frac{\partial}{\partial x^j}} Y = \nabla_{\frac{\partial}{\partial x^j}}\left(\eta^i \frac{\partial}{\partial x^i}\right) \quad \text{or} \quad \eta^i_{,j}\frac{\partial}{\partial x^i} = \frac{\partial \eta^i}{\partial x^j}\cdot\frac{\partial}{\partial x^i} + \eta^i\,\Gamma^k_{ji}\frac{\partial}{\partial x^i}$$

Changing the dummy indices $i \to k$, $k \to i$ in the 2nd term on the r. h. s we get

$$\eta^i_{.j}\frac{\partial}{\partial x^i} = \frac{\partial \eta^i}{\partial x^j}\cdot\frac{\partial}{\partial x^i} + \eta^k\,\Gamma^i_{jk}\frac{\partial}{\partial x^i}$$

Since $\left\{\frac{\partial}{\partial x^i} \;\therefore\; i = l,....,m\right\}$ is a basis and hence linearly independent and thus we must

have, $\eta^i_{.j} = \frac{\partial \eta^i}{\partial x^j} + \eta^k\,\Gamma^i_{jk}$

2. As ω is a tensor field of type (0. 1) we have from vi) of ξ.3.1

$$\left(\nabla_{\frac{\partial}{\partial x^i}}\omega\right)\left(\frac{\partial}{\partial x^k}\right) = \frac{\partial}{\partial x^i}\left(\omega\left(\frac{\partial}{\partial x^k}\right)\right) - \omega\left(\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^k}\right)$$

$$= \frac{\partial}{\partial x^i}\left(\omega_l d_x l\left(\frac{\partial}{\partial x^k}\right)\right) - \omega\left(\Gamma^h_{ik}\frac{\partial}{\partial x^h}\right)$$

$$= \frac{\partial}{\partial x^i}\left(\omega_l \delta^l_k\right) - \Gamma^h_{ik}\,\omega\left(\frac{\partial}{\partial x^h}\right)$$

$$\text{or, } \omega_{k,i} = \frac{\partial}{\partial x^i}(\omega_k) - \Gamma^h_{ik}\,\omega_h$$

$$\text{Thus, } \omega_{k,i} = \frac{\partial \omega_k}{\partial x^i} - \omega_h \Gamma^h_{ik}$$

3. From the definition

$$\left(\nabla_{\frac{\partial}{\partial x^k}} R\right)\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^m} = \nabla_{\frac{\partial}{\partial x^k}} R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^m} - R\left(\nabla_{\frac{\partial}{\partial x^k}}\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^m}$$

$$- R\left(\frac{\partial}{\partial x^i}, \nabla_{\frac{\partial}{\partial x^k}}\frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^m} - R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\nabla_{\frac{\partial}{\partial x^k}}\frac{\partial}{\partial x^m}$$

$$= \frac{\partial}{\partial x^k}\left(R^h_{ijm}\frac{\partial}{\partial x^h}\right) - R\left(\Gamma^s_{ki}\frac{\partial}{\partial x^s}, \frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^m} - R\left(\frac{\partial}{\partial x^i}, \Gamma^s_{kj}\frac{\partial}{\partial x^s}\right)\frac{\partial}{\partial x^m}$$

$$- R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\Gamma^s_{km}\frac{\partial}{\partial x^s}$$

$$= \left(\frac{\partial}{\partial x^k} R^h_{ijm}\right)\frac{\partial}{\partial x^h} + R^h_{ijm}\Gamma^s_{kh}\frac{\partial}{\partial x^s} - \Gamma^s_{ki}R^h_{sjm}\frac{\partial}{\partial x^h} - \Gamma^s_{kj}R^h_{ism}\frac{\partial}{\partial x^h} - \Gamma^s_{km}R^h_{ijs}\frac{\partial}{\partial x^h}$$

or $R^h_{ijm,k}\frac{\partial}{\partial x^h} = \left(\frac{\partial}{\partial x^k} R^h_{ijm}\right)\frac{\partial}{\partial x^h} + R^s_{ijm}\Gamma^h_{ks}\frac{\partial}{\partial x^h}$

$-\Gamma^s_{ki}R^h_{sjm}\frac{\partial}{\partial x^h} - \Gamma^s_{kj}R^h_{ism}\frac{\partial}{\partial x^h} - \Gamma^s_{km}R^h_{ijs}\frac{\partial}{\partial x^h}$, on changing the dummy indices $h \to s$, $s \to h$ in the 2nd term on the right hand side.

Since $\left\{\frac{\partial}{\partial x^h} \therefore h = l,\ldots,n\right\}$ is a basis and hence linearly independent and thus we must have,

$$R^h_{ijm.k} = \frac{\partial}{\partial x^k} R^h_{ijm} + R^s_{ijm}\Gamma^h_{sk} - R^h_{sjm}\Gamma^s_{ki} - R^h_{ism}\Gamma^s_{jk} - R^h_{ijs}\Gamma^s_{km}$$

UNIT - 4

§.4.1 Ricmannian Metric, Riemannian Connection :

Let us define a covariant tensor field of order 2 on M i.e. $g : \chi(M) \times \chi(M) \rightarrow F(M)$

Which satisfies

i) $g(X, X) > 0$: positive definite

ii) $g(X, X) = 0$ if and only if $X = \theta$: non singular

iii) $g(X,Y) = g(Y, X)$: symmetry , $\forall$ X, Y in $\chi(m)$

Such g is called a Riemannian metric on M and the differentiable manifold M together with such g is defined to be a **Riemannian Manifold, denoted symbolically by** (M, g).

Let $\{x^1, ; x^2, \ldots\ldots, x^n\}$ be a co-ordinate system is a neighbourhood U of $p \in M$. We define

iv) $g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = g_{ij}$

Note If we define $g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = \delta_{ij} = \begin{cases} 1, & i = j \\ o, & i \neq j \end{cases}$

then the matrix of g relative to the basis $\left\{\frac{\partial}{\partial x^i}\right\}$ is given by

$$g = \begin{pmatrix} 10\ldots\ldots 0 \\ 01\ldots\ldots 0 \\ \ldots\ldots\ldots\ldots \\ 00\ldots\ldots 1 \end{pmatrix}$$

A linear connection on a Riemanian manifold (M, g) is said to be a **metric connection** if and only if

4.1) $\nabla g = 0$ i.e. $(\nabla_X g)(Y, Z) = 0, \ \forall$ X, Y, Z in $\chi(M)$

The unique metric connection with vanishing torsion is called the **Riemannian Connection** or the **Levi-Civita Connection**. In this case

4.2) $\nabla_X Y - \nabla_Y X = [X, Y]$

Theorem 1 : Every Riemannian manifold (M, g) admits a unique Riemannian Connection.

Proof : To prove the existence of such a connection, let us define a mapping

$\nabla : \chi(M) \times \chi(M) \to \chi(M)$

denoted by

$\nabla : (X, Y) \to \nabla_X Y$

as follows

4.3) $2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) + g([X, Y], Z) + g(X,[Z, Y]) + g(Y,[Z, X])$

Clearly, $2g(\nabla_X Y + Z), W) - 2g(\nabla_X Y, W), 2g(\nabla_X Z, W)$

$= Xg(Y + Z, W) + (Y + Z)g(W, X) - Wg(X, Y + Z) + g([X, Y + Z], W) + g(X,[W, Y + Z])$

$+g(Y + Z,[W, X]) - Xg(Z, W) - Yg(W, X) + Wg(X, Z) - g([X, Y), W) - g(X,[W, Y])$

$-g(Y,[W, X]) - Xg(Z, W) - Zg(W, X) + Wg(X, Z) - g([X, Z], W)$

$-g(X,[W, Z]) - g(Z,[W, X])$

$= 0$

$\therefore\ 2g(\nabla_X (Y + Z) - \nabla_X Y - \nabla_X Z, W) = 0,$ as g is linear

Whence

$\nabla_X (Y + Z) = \nabla_X Y + \nabla_X Z$

Similarly it can be shown that

$\nabla_{X+Y} Z = \nabla_X Z + \nabla_Y Z,$

$\nabla_{f X} Y = f\, \nabla_X Y,$

$\nabla_X (fY) = (Xf)Y + f\, \nabla_X Y$

Thus such a mapping determines a linear connection on M. Also, from (4.3) it can be shown that

$2Xg(Y,Z) - 2g(\nabla_X Y,Z) - 2g(Y,\nabla_X Z) = 0$

or, $\nabla_X g(Y,Z) - g(\nabla_X Y,Z) - g(Y,\nabla_X Z) = 0$ by v) of ξ.3.1

or, $(\nabla_X g)(Y,Z) = 0, \ \forall X,Y,Z$

Thus such a linear connection admits a metric connection. Further, it can be shown that

$\nabla_X Y - \nabla_Y X - [X,Y] = 0$

Hence such a metric connection admits a Riemannian connection

To prove the uniqueness, let $\bar{\nabla}$ be another such connection. Then we must have

$Xg(Y,Z) - g(\nabla_X Y,Z) - g(Y,\nabla_X Z) = 0$ and $\nabla_X Y - \nabla_Y X - [X,Y] = 0$

$Xg(Y,Z) - g(\bar{\nabla}_X Y,Z) - g(Y,\bar{\nabla}_X Z) = 0$ and $\bar{\nabla}_X Y - \bar{\nabla}_Y X - [X,Y] = 0$

Subtracting,

$g(\bar{\nabla}_X Y - \nabla_X Y,Z) + g(Y,\bar{\nabla}_X Z - \nabla_X Z) = 0 \ \forall X,Y,Z$ and $\bar{\nabla}_X Y - \bar{\nabla}_X Y = \nabla_Y X - \nabla_Y X$

where form, we get

$\bar{\nabla}_X Y - \nabla_X Y = 0$

$\therefore \nabla_X Y = \bar{\nabla}_X Y$

Thus uniquences is established. This completes the proof

Exercise : 1 In terms of a local coordinate system $\{x^1, x^2, \cdots, x^n\}$ in a neighbourhood U of p of a Riemannian Manifold (M, g) show that

i) the components Γ^i_{jk} defined in UNIT 3 is symmetric and

ii) the Riemannian metric is covariantly constant.

2. Let ∇ be a metric connection of a Riemannian manifold (M, g) and $\tilde{\nabla}$ be another linear connecting given by

$\tilde{\nabla}_X Y = \nabla_X Y + T(X,Y)$

where T is the torsion tensor of M. Show that the following condition are equivalent

i) $\tilde{\nabla} g = 0$ and ii) $g(T(X,Y),Z) + g(Y,T(X,Y)) = 0$

3. In terms of a local coordinate system $\{x^1, \ldots\ldots, x^n\}$ the components Γ^i_{jk} of the Riemannian connection are given by

$$g_{im}\,\Gamma^i_{jk} = \frac{1}{2}\left(\frac{\partial g_{mk}}{\partial x^j} + \frac{\partial g_{jm}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^m}\right)$$

Solution : 1. A Riemannian Manifold (M, g) admits a unique Riemannian Connection i.e.

$$T = 0$$

$$\text{or } \nabla_X Y - \nabla_Y X - [X, Y] = 0$$

In terms of a local coordinate system $\{x^1, \ldots, x^n\}$, we have

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} - \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i} - \left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right] = 0$$

using 3.1),

$$\Gamma^k_{ij} \frac{\partial}{\partial x^k} - \Gamma^k_{ji} \frac{\partial}{\partial x^k} - 0 = 0$$

$$\text{or } \left(\Gamma^k_{ij} - \Gamma^k_{ji}\right) \frac{\partial}{\partial x^k} = 0$$

Since $\left\{\frac{\partial}{\partial x^k}\right\}$ is a basis and hence linearly independent and thus

$$\Gamma^k_{ij} = \Gamma^k_{ji} \text{ i.e. symmetric.}$$

By definition, on a Riemannian Manifold (M, g),

$$(\nabla_X g)\,(Y, Z) = 0, \ \forall\ X, Y, Z \text{ in } (M, g)$$

In terms of a local co-ordinate system $\{x^1, \ldots, x^n\}$, taking $X = \frac{\partial}{\partial x^i}$, $Y = \frac{\partial}{\partial x^j}$, $Z = \frac{\partial}{\partial x^k}$, we find

$$\left(\nabla_{\frac{\partial}{\partial x^i}} g\right)\left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\right) = 0$$

$$\text{or } \frac{\partial}{\partial x^i}\, g\left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\right) - g\left(\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\right) - g\left(\frac{\partial}{\partial x^j}, \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^k}\right) = 0$$

using 3.1) we get

$$\frac{\partial}{\partial x^i}\, g\left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\right) - g\left(\Gamma^l_{ij} \frac{\partial}{\partial x^l}, \frac{\partial}{\partial x^k}\right) - g\left(\frac{\partial}{\partial x^j}, \Gamma^l_{ik} \frac{\partial}{\partial x^l}\right) = 0$$

$$\text{or } \frac{\partial}{\partial x^i}\, g_{jk} - \Gamma^l_{ij}\, g_{lk} - \Gamma^l_{ik}\, g_{jl} = 0 \text{ by iv)}$$

$$\text{or, } g_{jk,i} = 0$$

i.e. Riemannian metric is covariantly constant.

2. Let us assume that i) be true. Then by definition,

$$Xg\,(Y, Z) - g\left(\tilde{\nabla}_X Y,\ Z\right) - g\left(Y,\ \tilde{\nabla}_X Z\right) = 0$$

Using the condition,

$$Xg(Y, Z) - g\left(\nabla_X Y + T(X,\ Y),\ Z\right) - g\left(Y,\ \nabla_X Z + T(X,\ Z)\right) = 0$$

$$\text{or } \left(\nabla_X g\ \right)(Y, Z) - g\left(T(X,\ Y),\ Z\right) - g\left(Y,\ T(X,\ Z)\right) = 0$$

Using 4.1), one gets

$$g\left(T(X,\ Y),\ Z\right) + g\left(Y,\ T(X,\ Z)\right) = 0$$

Let now the above result be true. Then using the condition

$$g\left(\tilde{\nabla}_X Y - \nabla_X Y,\ Z\right) + g\left(Y,\ \tilde{\nabla}_X Z - \nabla_X Z\right)$$

$$\text{or, } g\left(\tilde{\nabla}_X Y,\ Z\right) + g\left(Y,\ \tilde{\nabla}_X Z\ \right) = g(\nabla_X Y,\ Z) + g(Y,\ \nabla_X Z\ )$$

Using 4.1) on the right hand side we get

$$g\left(\tilde{\nabla}_X Y,\ Z\right) + g\left(Y,\ \tilde{\nabla}_X Z\ \right) = Xg(Y,\ Z)$$

$$\text{or, } g\left(\tilde{\nabla}_X Y,\ Z\right) + g\left(Y,\ \tilde{\nabla}_X Z\ \right) = \tilde{\nabla}_X g(Y,\ Z)$$

$$\text{i.e., } \left(\tilde{\nabla}_X g\right)(Y,\ Z) = 0 \ \forall\ X, Y, Z$$

$$\text{i.e., } \tilde{\nabla} g = 0$$

3. Using iv) we find $g_{im} = g\left(\dfrac{\partial}{\partial x^i},\ \dfrac{\partial}{\partial x^m}\right)$

$$\text{or, } 2g_{im}\,\Gamma^i_{jk} = 2g\left(\Gamma^i_{jk}\,\frac{\partial}{\partial x^i},\ \frac{\partial}{\partial x^m}\right)$$

Using 3.1) and 4.3) one gets the desired result ofter a few steps

Theorem 2 : If R is the curvature tensor of the Riemannian Manifold (M, g), then

4.4) $R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0$: Bianchi's 1st identity

4.5) $(\nabla_U R)\,(X, Y)Z + (\nabla_X R)\,(Y, U)Z + (\nabla_Y R)\,(U, X)Z = 0$: Bianchi's 2nd identity.

4.6) $g(X, Y)Z, U) = -g(R(X, Y)U, Z)$

4.7) $g(R(X, Y)Z, U) = -g(R(Z, U)X, Y)$

Proof : Using 3.3), 3.5) one gets

$R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = [X,[Y, Z]] + [Y,[Z, X]] + [Z,[X, Y]] = 0$ by Jacobi identity

4.5) is Left to the reader

To prove 4.6), one gets from 4.1)

$(\nabla_X g)(Z, U) = 0, \forall X, Z, U$

$\alpha) Xg(Z, U) = g(\nabla_X Z, U) + g(Z, \nabla_X U)$

or, $\nabla_Y (Xg(Z, U)) = \nabla_Y \{g(\nabla_X Z, U) + g(Z, \nabla_X U)\}$

or, $Y(Xg(Z, U)) = Yg(\nabla_X Z, U) + Yg(Z, \nabla_X U)$

using $\alpha)$ on the right side we get

$Y(Xg(Z, U) = g(\nabla_Y \nabla_X, Z, U) + g(\nabla_X Z, \nabla_Y U) + g(\nabla_Y Z, \nabla_Y U) + g(Z, \nabla_Y \nabla_X U)$

Thus, we find

$X(Yg(Z, U)) - Y(Xg(Z, U)) - [X, Y]g(Z, U)$

$= g\left(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla^Z_{[X,Y],} U\right) + g\left(Z, \nabla_X \nabla_Y U - \nabla_Y \nabla_X U - \nabla^U_{[X,Y]}\right)$

$= g(R(X, Y)Z, U) + g(Z, R(X, Y)U)$

Using the definition of [X, Y] f, on the left hand side, one finds

$g(R(X, Y)Z, U) + g(Z, R(X, Y)U) = 0$

Again, $R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0$

$g(R(X, Y)Z) + g(R(Y, Z)X, U) + g(R(Z, X)Y, U) = 0 \,.....\, \beta)$

Similarly, we can write

$g(R(U,Z)X,Y)+g(R(Z,X)U,Y)+g(R(X,U)Z,Y)=0$ ....γ)

$g(R(Y,X)U,Z)+g(R(X,U)Y,Z)+g(R(U,Y)X,Z)=0$ ....δ)

$g(R(Z,U)Y,X)+g(R(U,Y)Z,X)+g(R(Y,Z)U,X)=0$ ....ξ)

Adding $\alpha), \beta), \gamma), \delta), \xi)$ and using 4.6) we get

$g(R(X,Y)Z,U)+g(R(U,Z)X,Y)+g(R(Y,X)U,Z)+g(R(Z,U)Y,X)=0$

Using Exercise 3(ii) § 3.2 in the second and in the third term of the above equation.

or, $g(R(X,Y)Z,U)-g(R(Z,U)X,Y)-g(R(X,Y)U,Z)+g(R(Z,U)Y,X)=0$

After a few steps one gets

$$2g(R(X,Y)Z,U)=2g(R(Z,U)X,Y)$$

i.e. $g(R(X,Y)Z,U)+g(R(Z,U)X,Y)$

Exercise 4. In terms of a local coordinate system $\{x^1, \ldots\ldots, x^n\}$ in a neighbourhood U of p of (M, g) show that

i) $R^m_{ijk}+R^m_{jki}+R^m_{kij}=0$

ii) $R^h_{ijk,m}+R^h_{jmk,i}+R^h_{mik,j}=0$

iii) $R^h_{ijk}g_{hm}=-R^h_{jim}g_{hk}$

iv) $R^h_{ijk}g_{hm}=-R^h_{kmi}g_{hj}$

Solution : i) From ii) of Exercise 5 in § 3.2 and also using the result

$$\Gamma^m_{jk}=\Gamma^m_{kj}$$

the result follows immediately

ii) Left to the reader

ii) using ii) of Exercise 5 in § 3.2, on finds

$$R^h_{ijk}g_{hm}=\left(\frac{\partial}{\partial x^i}\Gamma^h_{jk}-\frac{\partial}{\partial x^i}\Gamma^h_{ik}+\Gamma^t_{jk}\Gamma^h_{ti}-\Gamma^t_{ik}\Gamma^h_{tj}\right)$$

$$=\frac{\partial}{\partial x^i}\left(\Gamma^h_{jk}g_{hm}\right)-\Gamma^h_{jk}\frac{\partial}{\partial x^i}\left(g_{hm}\right)-\frac{\partial}{\partial x^j}\left(\Gamma^h_{ik}g_{hm}\right)+\Gamma^h_{ik}\frac{\partial}{\partial x^j}g_{hm}$$

$$+\Gamma^t_{jk}\Gamma^h_{ti}\,g_{hm}-\Gamma^t_{ik}\Gamma^h_{tj}\,g_{hm}$$

Using Exercise 3 of § 4.1 we get

$$R^h_{ijk} g_{hm} = \frac{1}{2} \cdot \frac{\partial}{\partial x^i} \left(\frac{\partial g_{mk}}{\partial x^j} + \frac{\partial g_{mj}}{\partial x^k} + \frac{\partial g_{jk}}{\partial x^m} \right) - \Gamma^h_{jk} \frac{\partial g_{mh}}{\partial x^i}$$

$$-\frac{1}{2} \frac{\partial}{\partial x^j} \left(\frac{\partial g_{mi}}{\partial x^k} + \frac{\partial g_{mk}}{\partial x^i} - \frac{\partial g_{ik}}{\partial x^m} \right) + \Gamma^h_{ik} \frac{\partial g_{hm}}{\partial x^j} + \frac{1}{2} \Gamma^t_{jk} \left(\frac{\partial g_{mt}}{\partial x^i} + \frac{\partial g_{mi}}{\partial x^t} - \frac{\partial g_{ti}}{\partial x^m} \right)$$

$$-\frac{1}{2} \Gamma^t_{ik} \left(\frac{\partial g_{mt}}{\partial x^j} + \frac{\partial g_{mj}}{\partial x^t} - \frac{\partial g_{tj}}{\partial x^m} \right)$$

Similarly, one can write $R^h_{ijm} g_{hk}$

Thus, $R^h_{ijk} g_{hm} + R^h_{ijm} g_{hk}$

$$= -\frac{1}{2} \Gamma^h_{jk} \left(\frac{\partial g_{hm}}{\partial x^i} + \frac{\partial g_{hi}}{\partial x^m} - \frac{\partial g_{mi}}{\partial x^h} \right) + \frac{1}{2} \Gamma^h_{ik} \left(\frac{\partial g_{hm}}{\partial x^j} + \frac{\partial g_{hj}}{\partial x^m} - \frac{\partial g_{mj}}{\partial x^h} \right)$$

$$-\frac{1}{2} \Gamma^h_{jm} \left(\frac{\partial g_{kh}}{\partial x^i} + \frac{\partial g_{hi}}{\partial x^k} - \frac{\partial g_{ik}}{\partial x^h} \right) + \frac{1}{2} \Gamma^h_{im} \left(\frac{\partial g_{hk}}{\partial x^j} + \frac{\partial g_{hj}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^h} \right)$$

$$= -\Gamma^h_{jk} \Gamma^t_{im} g_{th} + \Gamma^h_{ik} \Gamma^t_{jm} g_{th} - \Gamma^h_{jm} \Gamma^t_{ik} g_{th} + \Gamma^h_{im} \Gamma^t_{jk} g_{th}$$

Thus, $R^h_{ijk} g_{hm} + R^h_{ijm} g_{hk} = 0$ or $R^h_{ijk} g_{hm} = -R^h_{ijm} g_{hk}$

iv) From Exercise iii) above we write

$$R^h_{ijk} g_{hm} - R^h_{kmi} g_{hj} = -\frac{1}{2} \Gamma^h_{jk} \frac{\partial g_{mh}}{\partial x^i} + \frac{1}{2} \Gamma^h_{jk} \frac{\partial g_{mi}}{\partial x^h} - \frac{1}{2} \Gamma^h_{jk} \frac{\partial g_{hi}}{\partial x^m}$$

$$+\frac{1}{2} \Gamma^h_{mi} \frac{\partial g_{jh}}{\partial x^k} - \frac{1}{2} \Gamma^h_{mi} \frac{\partial g_{jk}}{\partial x^h} + \frac{1}{2} \Gamma^h_{mi} \frac{\partial g_{hk}}{\partial x^j}$$

$$= -\frac{1}{2} \Gamma^h_{jk} \left(\frac{\partial g_{mh}}{\partial x^i} + \frac{\partial g_{hi}}{\partial x^m} - \frac{\partial g_{mi}}{\partial x^h} \right) + \frac{1}{2} \Gamma^h_{mi} \left(\frac{\partial g_{hj}}{\partial x^k} + \frac{\partial g_{hk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^h} \right)$$

$$= -\frac{1}{2} \Gamma^h_{jk} \Gamma^t_{im} g_{th} + \frac{1}{2} \Gamma^h_{mi} \Gamma^t_{jk} g_{th} = 0 \quad \therefore\ R^h_{ijk} g_{hm} = R^h_{kmi} g_{hj}$$

Theorem 3 : If $\overline{\nabla}$ and ∇ correspond to the Levi-Civita (Riemannian) Connection and the metric connection with non-vanishing torsion T, then

4.8) $\nabla_X Y - \overline{\nabla}_X Y = \frac{1}{2}\{T(X, Y) + T'(X, Y), T'(Y, X)\}$ where

4.9) $g(T(Z, X), Y) = g(T'(X, Y), Z)$

Proof : From 4.1) we see that

$$(\nabla_X g)(Y, Z) = 0 \text{ and } (\overline{\nabla}_X g)(Y, Z) = 0$$

Thus $$Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \text{ and}$$

$$Xg(Y, Z) = g(\overline{\nabla}_X Y, Z) + g(Y, \overline{\nabla}_X Z)$$

Subtracting these two, we get

$$\text{a)} \quad \begin{cases} g\big(U(X, Y), Z\big) + g\big(Y, U(X, Z)\big) = 0 \quad \text{where} \\ U(X, Y) = \overline{\nabla}_X Y - \nabla_X Y, \\ U(X, Z) = \overline{\nabla}_X Z - \nabla_X Z \end{cases}$$

Again from 4.2) we get

$$0 = \overline{\nabla}_X Y - \overline{\nabla}_Y X - [X, Y] \text{ and}$$

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$$

Subtracting and using a) above

$$-T(X, Y) = U(X, Y) - U(Y, X)$$

$$\text{or, } g(T(X, Y), Z) = g(U(Y, X), Z) - g(U(X, Y), Z)$$

Again, on using 4.9), we find

$$\begin{aligned} g(T(X, Y), Z) + g(T'(X, Y), Z) + g(T'(Y, X), Z) &= g(T(X, Y), Z) \\ &\quad + g(T(Z, X), Y) + g(T(Z, Y), X) \\ &= g(U(Y, X), Z) - g(U(X, Y), Z) + g(U(X, Z), Y) - g(U(Z, X), Y) \\ &\quad + g(U(Y, Z), X) - g(U(Z, Y), X) \\ &= -2g(U(X, Y), Z) \quad \text{by a)} \\ &= -2g(\overline{\nabla}_X Y - \nabla_X Y, Z) = 2g(\nabla_X Y - \overline{\nabla}_X Y, Z) \end{aligned}$$

$$\therefore \nabla_X Y - \overline{\nabla}_X Y = \frac{1}{2}\{T(X, Y) + T'(X, Y), T'(Y, X)\}$$

3.4.2 Riemann Curvature tensor field :

The Riemann Curvature tensor field of 1st kind of M is a tensor field of degree (0, 4), denoted also by R

$R : \chi(M)\times\chi(M)\times\chi(M)\times\chi(M) \to F(M)$

and defined by

4.10) $R(X,Y,Z,W) = g(R(X,Y)Z,W), X,Y,Z,W$ in $\chi(M)$

Exercise : 1 Verify that

i) R(X, Y, Z, W) = – R(Y, X, Z, W)

ii) R(X, Y, Z, W) = – R(X, Y, W, Z)

iii) R(X, Y, Z, W) = – R(Z, W, X, Y)

iv) R(X, Y, Z, W) + R(Y, Z, X, W) + R(Z, X, Y, W) = 0

v) $(\nabla_U R)(X,Y,Z,W) + (\nabla_Z R)(X,Y,W,U) + (\nabla_W R)(X,Y,U,Z) = 0$

2. If R^h_{ijk} and g_{hm} are the components of the curvature tensor and the metric tensor with respect to a local coordinate system $x^1, x^2, \cdots\cdots, x^n$ then the components R_{ijkm} of the Rieman Curvature tensor are given by

$$R_{ijkm} = R^h_{ijk}\, g_{hm}$$

where $R_{ijkm} = \left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^m}\right)$

3. A vector field z on (M, g) is called a **gradient vector field** if

4.11) $$g(Z,Y) = df(Y) = Yf,\ f \in F(M)$$

for every vector field Y and M. Show that for such Z

$g(\nabla_X Z, Y) = g(\nabla_Y Z, X)$ for every vector field X on M.

Solution : From 4.1) we see that

$$(\nabla_X g)(Y,Z) = 0 \text{ for all X, Y, Z in } \chi(M)$$

$$\text{or } Xg(Y,Z) - g(\nabla_X Y, Z) = g(Y, \nabla_X Z)$$

Using 4.11), one finds

$$g(\nabla_X Z, Y) = X(Yf) - g(\nabla_X Y, Z)$$

similarly $$g(\nabla_Y Z, X) = Y(Xf) - g(\nabla_Y X, Z)$$

$\therefore\ g(\nabla_X Z, Y) - g(\nabla_Y Z, X) = X(Yf) - Y(Xf) + g(\nabla_Y X, Z) - g(\nabla_X Y, Z)$

or, $g(\nabla_X Z, Y) - g(\nabla_Y Z, X) = [X, Y]f - g(\nabla_X Y - \nabla_Y X, Z)$

$= [X, Y]f - g[X, Y], Z)$ by 4.2)

$= [X, Y]f - [X, Y]f$ by 4.11)

$= 0$

Thus

$$g(\nabla_X Z, Y) - g(\nabla_Y Z, X)$$

3.4.3 Einstein Manifold :

Let $\{e_1, e_2, \cdots e_n\}$ be an orthonormal basis of $T_p(M)$ Then the Ricci tensor field, denoted by S, is the covariant tensor field of degree 2 and is defined by

$$S(X_p, Y_p) = \sum_{i=1}^{n} R\left(((e_i)_P, X_P, Y_P, (e_i)_P\right)$$

We write it as

4.12) $$S(X, Y) = \sum_{i=1}^{n} R(e_i, X, Y, e_i)$$

Such a tensor field S(X, Y) is also called the **Ricci Curvature** of M.

If there is a constant λ such that

4.13) $S(X, Y) = \lambda g(X, Y)$

then M is called on **Einstein Manifold.**

The function r on M, defined by

$$r(p) = \sum_{i=1}^{n} S\left((e_i)_P, (e_i)_P\right)$$

is called the **scalar curvature** of M. We write it as

4.14) $$r = \sum_{i=1}^{n} S(e_i, e_i)$$

Exercise : 1. Show that the Ricci tensor field is symmetric.

At any $p \in M$, we denoted by Π a plane section i.e., a two dimensional subspace of $T_p(M)$. The sectional curvature of Π denoted by K(Π) with orthonormal basis X, Y is defined as

4.15) K(Π)=g(R(X, Y) Y, X) = R(X, Y, Y, X)

If K(Π) is constant for all plane section and for all points of p M, Then (M, g) is called a manifold of constant curvature. For such a manifold

4.16) $R(X,Y)Z = k\{g(Y,Z)X - g(X,Z)Y\}$ where $k(\Pi)$ say

Example : Euclidean space is of Constant Curvature

Exercise : 1, Show that a Riemannian manifold of constant curvature is an Einstein Manifold.

2. If M is a 3-dimensional Einstein Manifold, then, it is a manifold of constant curvature

Solution : Let $\{X_1, X_2, X_3\}$ be an orthonormal basis of $T_p(M)$ Then, the sectional curvature with orthonormal basis X_1, X_2 denoted by $K(\Pi_{12})$ is given by

$$K(\Pi_{12}) = R(X_1, X_2, X_2, X_1)$$

$$= R(X_2, X_1, X_1, X_2)$$

$$= K(\Pi_{21})$$

Thus, $K(\Pi_{ij}) = K(\Pi_{ji}), i \neq j$

Again from 4.12)

$$S(X_1, X_2) = \sum_{i=1}^{3} R(X_i, X_1, X_2, X_i)$$

$$= R(X_1, X_1, X_2, X_1) + R(X_2, X_1, X_2, X_1) + R(X_3, X_1, X_2, X_3)$$

$$= 0 + K(\Pi_{21}) + K(\Pi_{31})$$

$$= K(\Pi_{12}) + K(\Pi_{13})$$

$S(X_2, X_2) = K(\Pi_{21}) + K(\Pi_{23})$ and

$S(X_3, X_3) = K(\Pi_{31}) + K(\Pi_{32})$

As it is a 3-dimensional Einstein manifold, so from 4.13)

$$S(X_1, X_1) = \lambda g(X_1, X_1) = \lambda$$

$$S(X_1, X_2) = \lambda g(X_1, X_2) = 0$$

Thus, $S(X_1, X_1) + S(X_2, X_2) - S(X_3, X_3) = 2K(\Pi_{12})$

or, $\lambda = 2K(\Pi_{12})$

$$\therefore\ K(\Pi_{12}) = \frac{\lambda}{2} = \text{constant.}$$

i.e. $K(\Pi_{ij}) = \text{Constant}, \ i \neq j$

Thus every 3 deminsional Einstein manifold is a manifold of constant curvature.

§ 4.4 Semi-symmetric Metric Connection

A linear connection is said to be a semi-symmetric connection if

4.17) $T(X,Y) = w(Y)X - w(X)Y$, for every 1-form w.

A linear connection for which

4.18) $\nabla g = 0$

is called a semi-symmetric metric connection.

Theorem 1 : If ∇ and $\overline{\nabla}$ correspond to semi-symmetric connection and Levi-Civita Connection respectively, then,

$$\nabla_X Y - \overline{\nabla}_X Y = w(Y)X - g(X, Y)p$$

Where p is a vector field given by

$$g(X, p) = w(X)$$

Proof : Since ∇ correspond to a semi-symmetric connection, by 4.17)

$$T(Z, X) = w(X)Z - w(Z)X$$

$$g(T(Z, X), Y) = g(w(X)Z - w(Z)X, Y)$$

$$= w(X)\, g(Z, Y) - w(Z)\, g(X, Y)$$

Using Theorem 3 of § 4.1 on the l. h. s. we get

$$g(T'(X, Y), Z) = w(X)\, g(Y, Z) - g(Z, p)\, g(X, Y)$$

$$= g(w(X)Y, Z) - g(Z, g(X, Y)\, p)$$

$$= g(w(X)Y - g(X, Y)p, Z)$$

Whence $T'(X, Y) = w(X)Y - g(X, Y)p$

using the above result in 4.8) we get

$$\nabla_X Y - \bar{\bar{\nabla}}_Y Y = \frac{1}{2}\{T(X,Y) + \omega(X)Y - g(X,Y)p + \omega(Y)X - g(Y,X)p\}$$

Again using 4.17), one gets

$$\nabla_X Y - \bar{\bar{\nabla}}_Y Y = \omega(Y)X - g(X,Y)p$$

Exercise 1. If ∇ and $\bar{\bar{\nabla}}$ correspond to a semi-symmetric connection and the Levi-Civita connection respectively, then for any 1-form ω

$$(\nabla_X \omega) = \left(\bar{\bar{\nabla}}_X \omega\right) Y - \omega(X)\omega(Y) + \omega(p)g(X,Y), \text{ where}$$

$$g(X,p) = \omega(X)$$

2. Let $\bar{\bar{\nabla}}$ be the Levi-Civita Connection and ∇ be another linear connection such that

$$\nabla_X Y = \bar{\bar{\nabla}}_X Y - \omega(X)Y \text{ where is a 1-form.}$$

Show that ∇ is a semi-symmetric connection for which $\nabla_X g = 2\omega(X)g$

Hints : 1. Note that

$$(\nabla_X \omega)Y = X\omega(Y) - \omega(\nabla_X Y)$$

Use Theorem 1 in the second term on the right hand side, one gets the desired result.

2. Note that

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$$

$$= \bar{\bar{\nabla}}_X Y - \omega(X)Y - \bar{\bar{\nabla}}_Y X + \omega(Y)X - [X,Y]$$

$$= \bar{\bar{T}}(X,Y) + \omega(Y)X - \omega(X)Y, \text{ on using the hypothesis}$$

$$\omega(Y)X - \omega(X)Y, \text{ as } \bar{\bar{T}} = 0.$$

Again,

$$(\nabla_X g)(Y,Z) = Xg(Y,Z) - g(\nabla_X Y,Z) - g(Y,\nabla_X,Z)$$

$$= Xg(Y,Z) - g(\bar{\bar{\nabla}}_X Y - \omega(X)Y,Z) - g(Y,\bar{\bar{\nabla}}_X Z - \omega(X)Z)$$

$$= (\bar{\bar{\nabla}}_X g)(Y,Z) + 2\omega(X)g(Y,Z), \text{ on using the hypothesis}$$

$$\therefore\ \nabla_X g = 2\omega(X)g, \text{ as } \bar{\bar{\nabla}}g = 0.$$

§ 4.5 Weyl Conformal Curvature tensor :

The Weyl conformal curvature tensor, denoted by C, is defined on an n-dimensional Riemannian manifold (M, g) as follows :

$$
4.19)\quad \begin{cases} C(X, Y)Z = R(X, Y)Z + \lambda(Y, Z)X - \lambda(X, Z)Y + g(Y, Z)LX - g(X, Z)LY \\ \text{where } \lambda \text{ is defined by} \\ \lambda(X, Y) = -\dfrac{1}{n-2}\, S(X, Y) + \dfrac{r}{2(n-1)(n-2)}\, g(X, Y) \text{ and L is a tensor field of} \\ \text{type (1, 1) given by} \\ g(LX, Y) = \lambda(X, Y), \text{ for every vector field X, Y, Z on M} \end{cases}
$$

An n-dimensional ($n > 3$) Riemannian manifold is said to be conformally flat if

4.20) $\qquad C(X, Y)Z = 0$

Goldberg's Result :

Let (M, g) be a Riemannian manifold and A be the field of symmetric endomorphism corresponding to the Ricci tensor S i.e.

4.21) $g(AX, Y) = S(X, Y)$ for every vector fields X, Y on M. Then

$$
4.22)\ C(X, Y)Z = R(X, Y)Z - \frac{1}{n-2}\{g(Y, Z)AX - g(X, Z)AY + S(Y, Z)X - S(X, Z)Y\}
$$

$$
+ \frac{r}{(n-1)(n-2)}\{g(Y, Z)X - g(X, Z)Y\}
$$

Proof : Note that

$$
g(g(Y, Z)LX, Y) = g(Y, Z)\, g(LX, Y) = g(Y, Z)\, \lambda(X, Y) \text{ by 4.19)}
$$

$$
= -\frac{1}{n-2}\, g(Y, Z)\, S(X, Y) + \frac{rg(Y, Z)}{2(n-1)(n-2)}\, g(X, Y) \text{ by 4.19)}
$$

$$
= -\frac{1}{n-2}\, g(Y, Z)\, g(AX, Y) + \frac{rg(Y, Z)}{2(n-1)(n-2)}\, g(X, Y) \text{ by 4.21)}
$$

$$
\text{or } g(Y, Z)LX = -\frac{g(Y, Z)}{(n-2)}AX + \frac{rg(Y, Z)}{2(n-1)(n-2)}X
$$

Using the above result & 4.19) we find

$$
C(X, Y)Z = R(X, Y)Z - \frac{1}{n-2}\, S(Y, Z)X + \frac{rg(Y, Z)}{2(n-1)(n-2)}X + \frac{1}{n-2}\, S(X, Z)Y
$$

$$
- \frac{rg(X, Z)Y}{2(n-1)(n-2)} - \frac{g(Y, Z)AX}{n-2} + \frac{rg(Y, Z)X}{2(n-1)(n-2)} + \frac{g(X, Z)}{n-2}AY - \frac{rg(X, Z)Y}{2(n-1)(n-2)}
$$

Or, $C(X,Y)Z = R(X,Y)Z - \frac{1}{n-2}\{g(Y,Z)AX - g(X,Z)AY + S(Y,Z)X - S(X,Z)Y\}$

$$+\frac{r}{(n-1)(n-2)}\{g(Y,Z)X - g(X,Z)Y\}$$

Exercise : 1 If an $n(n>3)$ – dimensional Einstein Manifold is conformally flat than

2. If we write

$$R_{ijkl} = R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l}\right)$$

$$C_{ijkl} = g\left(C\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l}\right)$$

$$R_{ij} = S\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)$$

show that

$$C_{ijkl} = R_{ijkl} - \frac{1}{n-2}\left\{g_{jk}R_{il} - g_{ik}R_{jl} + R_{jk}g_{il} - R_{ik}g_{jl}\right\}$$

$$+\frac{r}{(n-1)(n-2)}\left\{g_{jk}g_{il} - g_{ik}g_{jl}\right\}$$

Hints : 1 Using 4.13) in 4.14, one gets $r = \lambda n$

Alsing above result, 4.13), one gets from 4.21)

$$Ax = \frac{r}{n}x$$

Using 4.20) in 4.22) and also the result deduced above, one gets the desired result after a few steps.

2. Using goldberg's result, one gets from the hypothesis

$$C_{ijkl} = g\left(C\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l}\right)$$

the desired result.

4.5 Conformally Symmetric Riemannian Manifold :

A Riemannian manifold (M, g) is said to be conformally symmetric if

4.23) $\nabla C = 0$

Where C is the Weyl Conformal Curvature tensor

Theorem 1 : A conformally symmetric manifold is of constant scalar curvature if

$$(\nabla_Z S)(Y, W) = (\nabla_W S)(Y, Z) \text{ for all Y, Z, W}$$

Proof : From 4.22) we see that

$$C(X,Y,Z,W) = R(X,Y,Z,W) - \frac{1}{n-2}\{g(Y,Z)g(AX,W) - g(X,Z)g(AY,W) +$$

$$+S(Y,Z)g(X,W) - S(X,Z)g(Y,W)\} + \frac{r}{(n-1)(n-2)}\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\}$$

Taking co-variant derivative on both sides and using (4.23), we get

$$\therefore\ (\nabla_U R)(X,Y,Z,W) = \frac{1}{n-2}\{g(Y,Z)(\nabla_U S)g(X,W) - g(X,Z)(\nabla_U S)g(Y,W)$$

$$+(\nabla_U S)(Y,Z)g(X,W) - (\nabla_U S)(X,Z)g(Y,W)\}$$

$$-\frac{\nabla_U r}{(n-1)(n-2)}\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\}$$

It is known from Exercise 1(v) of ξ 4.2 that

$$(\nabla_U R)(X,Y,Z,W) + (\nabla_U R)(X,Y,W,U) + (\nabla_W R)(X,Y,U,Z) = 0$$

Using the result deduced above, and also the hypothesis one gets

$$\nabla_U r\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\} + \nabla_Z r\{g(Y,W)g(X,U) - g(X,W)g(Y,U)\}$$

$$+\nabla_W r\{g(Y,U)g(X,Z) - g(X,U)g(Y,Z)\} = 0$$

Let $\{e_i : i = 1, \cdots\cdots, n\}$ be an orthonormal basis vectors.

Taking the sum for $1 \le i \le n$ for $X = U = e_i$, we get on using the result

$$\nabla_{ei} r g(e_i, z) = \nabla_z r$$

that

$$g(Y,Z)\nabla_w r - g(Y,W)\nabla_z r + ng(Y,W)\nabla_z r - g(Y,W)\nabla_z r + g(Y,Z)\nabla_w r - ng(Y,Z)\nabla_w r = 0$$

or $g(Y,Z)\nabla_w r - g(Y,W)\nabla_z r = 0$

Finally taking the sum for $1 \le i \le n$ for $Y = Z = e_i$, we get

$$\nabla_w r = 0, \ n > 1.$$

Thus the manifold is of constant curvature.

Definition : A linear transformation A is symmetric or skew symmetric according as

4.24) $$\begin{cases} g(AX, Y) = g(X, AY) \\ \text{or} \\ g(AX, Y) = -g(X, AY) \end{cases}$$

Exercise : 1. Show that for a symmetric linear transformation A and a skew-symmetric linear transformation $\bar{R}$, the new linear transformation T defined by, $T = A. \ \bar{R} = \bar{R}. \ A$ is skew - symmetric.

Theorem 2 : For a conformally flat $n(n > 3)$ - dimensional Riemannian manifold, the curvature tensor R is of the form

$$R(X,Y) = \frac{1}{n-2}(AX \wedge Y + X \wedge AY) - \frac{r}{(n-1)(n-2)} X \wedge Y$$

where $X \wedge Y$ denotes the skew - symmetric endomarphism of the tangent space at every point defined by

$$(X \wedge Y)Z = g(Y,Z)X - g(X,Z)Y$$

Proof : Using the hypothesis, we find that

$$(AX \wedge Y)Z + (X \wedge AY) = g(Y,Z)AX - g(X,Z)AY + S(Y,Z)X - S(X,Z)Y$$

As the manifold is conformally flat, we get on using the above result and the hypothesis,

$$R(X,Y)Z = \frac{1}{n-2}\{(AX \wedge Y)Z + (X \wedge AY)Z\} - \frac{r}{(n-1)(n-2)}\{X \wedge Y)Z\}$$

i.e. $R(X,Y) = \frac{1}{n-2}(AX \wedge Y + X \wedge AY) - \frac{r}{(n-1)(n-2)} X \wedge Y$

Theorem 3 : If in a conformally flat manifold, for a symmetric linear transformation A,

$$R(X, Y)A = A.\ R\ (X, Y)$$

then

$$\left(A^2 - \frac{rA}{n-1}\right) X \wedge X = 0$$

Proof : Note that

$$R(X, Y) = - R(Y, X)$$

As A is symmetric, so by Exercise 1 of this article A. R(X, Y) = R(X, Y). A is skew - symmetric. Thus R(Z, W)A is a skew symmetric linear transformation and from 4.24) we can write

$$g((R(Z, W)A)X, X) = - g(X, (R\ (Z, W)\ A)\ X)$$

or $$g(R(Z, W)A)X, X) = - g(X, R\ (Z, W)\ AX)$$

$$= - g(R(Z, W)\ AX, X), \text{ as g is symmetric.}$$

$$\therefore\ g(R(Z, W)AX, X) = 0$$

Using 4.7) one gets

$$g(R\ (AX, X)Z, W) = 0$$

Whence $$R(AX, X)Z = 0$$

i.e., $$R(AX, X) = 0$$

Again $(AX \wedge AX)Z = 0$ i.e., $AX \wedge AX = 0$ for every Z.

Using Theorem 2, one gets

$$R(X,AX) = \frac{1}{n-2}(AX \wedge AX + X \wedge A^2X) - \frac{r}{(n-1)(n-2)} X \wedge AX$$

AS R(AX, X) = –R(X, AX) and R(AX, X) = 0, we get from above,

$$X \wedge A^2X - \frac{r}{n-1} X \wedge AX = 0$$

Note that $X \wedge Y$ is skew - symmetric and thus

$$A^2X \wedge X - \frac{r}{n-1} AX \wedge X = 0$$

$$\therefore \left(A^2 - \frac{r}{n-1}\right) X \wedge X = 0$$

Definition : A curve $\sigma = x(t), a \leq t \leq b$ is called a geodesic on M with a linear connection ∇ if

4.25) $\nabla_X X = 0$

Where X is the vector tangent to the integral curve σ at x(t). Note that the integral curves of a left invariant vector fields are geodesic.

4.7 Biinvariant Riemannian metric on a Lie group :

A Riemannian metric g on a Lie group is said to be biinvariant if it is both left and right invariants.

Exercise 1 : If g is a left invariant convariant tensor field of order 2 on G and X, Y are left invariant vector fields on G, show that g(X, Y) is a constant function.

Theoxem 1 : If G is a Lie group admitting a biinvariant Riemannian metric g, then

4.26) g([X, Y], Z) = g(X, [Y, Z])

4.27) $R(X,Y)Z = -\frac{1}{4}[[X,Y],Z]$

4.28) $g(R(X,Y)Z, W) = -\frac{1}{4} g([X,Y],[Z,W])$

Proof : Since X, Y are left invariant vector fields, X + Y is also so and hence from 4.25)

$$\nabla_{X+Y}^{X+Y} = 0$$

Using 4.25, we find from above

i) $$\nabla_X Y + \nabla_Y X = 0$$

since M admits a unique Riemannian connection, we must have

$$\nabla_X Y - \nabla_Y X - [X, Y] = 0$$

ii) or $\nabla_X Y = \frac{1}{2}[X, Y]$ from i)

Now for a Riemannian Manifold $(\nabla_Y g)(X, Z) = 0$

or, $$Yg(X, Z) - g(\nabla_Y X, Y) - g(X, \nabla_Y Z) = 0$$

Using Exercise 1 of this article and Exercise 2 of ξ 1.4 we see that

$$Y.\ g(X, Z) = 0$$

Thus from ii) we find that $-\frac{1}{2}g([Y, X]Z) - \frac{1}{2}g(X, [Y, Z]) = 0$

or, $$g([X, Y], Z) - g(X, [Y, Z])$$

or, $$g([X, Y], Z) = g(X, [Y, Z])$$

Again from the definition

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla^Z_{[X,Y]}$$

$$= \frac{1}{4}[X,[Y,Z]] - \frac{1}{4}[Y,[X,Z]] - \frac{1}{2}[[X, Y], Z] \text{ by using ii)}$$

$$= \frac{1}{4}[X,[Y,Z]] + \frac{1}{4}[Y,[X,Z]] - \frac{1}{2}[[X, Y], Z]$$

$$= -\frac{1}{4}[Z,[X, Y]] - \frac{1}{2}[[X, Y], Z] \text{ by Jacobi Identity}$$

$$= \frac{1}{4}[[X, Y], Z] - \frac{1}{2}[[X, Y], Z]$$

$$= -\frac{1}{4}[[X, Y], Z]$$

Again $R(X, Y)Z, W) = -\frac{1}{4}g\left([[X, Y], Z],\ W\right)$ by 4.27)

$$= -\frac{1}{4}g\left([X, Y], Z],\ [Z, W]\right) \text{ by 4.26)}$$

This completes the proof.

Theorem 2 : If G is a Lie group admitting a biinvariant Riemannian metric g and Π is a plane section in $T_p(M)$ where Π is determined by orthonormal left invariant vector fields X, Y at p on G, then the sectional curvature at p is zero if and only if [X, Y] = 0.

Proof : From 4.15)

$$K(\Pi) = g(R(X,Y,)Y,X)$$

$$= -\frac{1}{4}g([X,Y],[Y,X]) \text{ by 4.28)}$$

$$= \frac{1}{4}g([X,Y],[X,Y])$$

The result follows immediately as g is nonsingular.

Theorem 3 : If G is a Lie group admitting a biinvariant Riemannian metric g, then

for all left invariant vector fields, X, Y, Z, W, P.

Proof : From Jacobi's identity

$$[W, [P, Z]] + [P, [Z, W]] + [Z, [W, P]] = 0$$

Taking P = [X, Y], we get

$[W, [[X, Y], Z] + [[X, Y], [Z, W]] + [Z, [W, [X, Y]]] = 0$

or $[W, [[X, Y], Z]] - [[X, Y], [W, Z]] = [[W, [X, Y]], Z]$

$= [- [X, [Y, W]] - [Y, [W, X]], Z]$ by Jacobi Identity

i) $[W, [[X, Y], Z]] - [[X, Y], [W, Z]] = [[X, [W, Y]], Z] + [[W, X], Y], Z]$

Again from the definition

$$(\nabla_W R)(P,Z,X,Y) = \nabla_W R(P,Z,X,Y) - R(\nabla_W P,Z,X,Y) - R(P,\nabla_W Z,X,Y) -$$

$$-R(P,Z,\nabla_W X,Y) - R(P,Z,X,\nabla_W Y)$$

$$= 0 + R(X,Y,Z,\nabla_W P) + R(X,Y,\nabla_W Z,P) + R(\nabla_W X,Y,Z,P)$$

$$+P(X,\nabla_W Y,Z,P)$$

Using 4.28), one gets

$$(\nabla_W R)(P,Z,X,Y) = -\frac{1}{8} g\big([X,Y],[Z,[W,P]]\big) - \frac{1}{8} g\big([[W,Z],P],[X,Y]\big)$$

$$-\frac{1}{8} g\big([[W,X]Y][Z,P]\big) - \frac{1}{8} g\big([X,[W,Y],[Z,P]]\big)$$

Using 4.26) successively we get

$$= -\frac{1}{8}\Big\{g\big(\big[[[X,Y],Z],W\big],P\big) + g\big([[X,Y],[W,Z]],P\big)$$

$$+g\big([[W,X],Z],P\big) + g\big([[X,[W,Y]],P]\big)\Big\}$$

$$= +\frac{1}{8} g\big(\big[W,[[X,Y],Z\big],P\big) - \frac{1}{8} g\big([[X,Y][W,Z]],P\big)$$

$$-\frac{1}{8} g\big([[X,[W,Y]],Z],P\big) - \frac{1}{8} g\big([[[W,X],Y],Z],P\big)$$

= 0 by i) for all left invariant vector fields X, Z, Y, W, P.

This completes the proof.

REFERENCES

1. W. B. Boothby : An Introduction to differentiable Manifold and Riemannian Geometry.

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